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Anders Källström

Mathematics for Computer Graphics Third meeting November 2003

Anders Källström

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Projections and perspective

Three-dimensional objects cannot be drawn on a computer screen. We have to *project* the object on a *viewing plane*. This is an operation that is easily described using homogeneous coordinates.

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Projections and perspective

Three-dimensional objects cannot be drawn on a computer screen. We have to *project* the object on a *viewing plane*. This is an operation that is easily described using homogeneous coordinates.

Let the position of the eye be given by the vector $\langle E|$, the point we are viewing by $\langle P|$ and the image in the plane with equation $\langle X|N\rangle = 0$ by the point $\langle P'|$.

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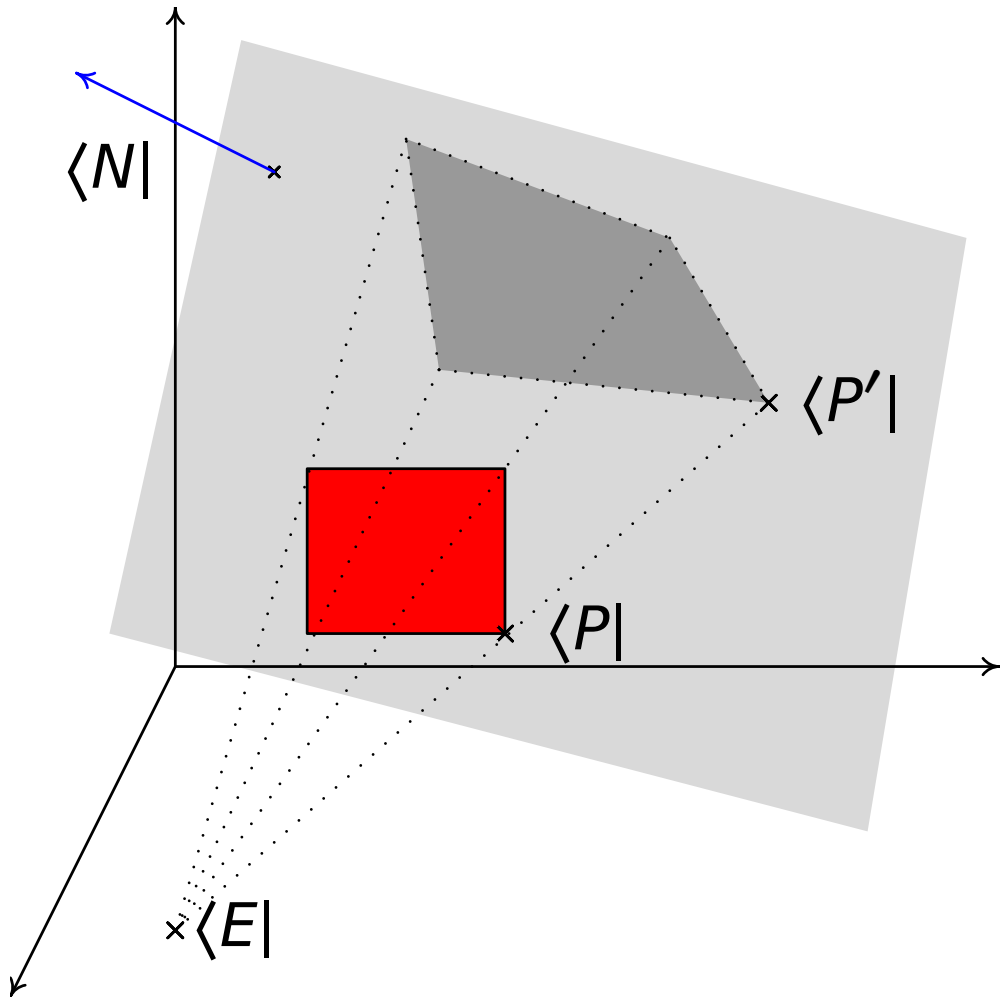

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The point $\langle P'|$ is obtained as the intersection between the line through $\langle E|$ and $\langle P|$ and the viewing plane. Hence we get the following equations

$$\langle X| = \langle E| + t\big(\langle P| - \langle E|\big) \tag{1}$$

$$\langle X|N\rangle = 0 \tag{2}$$

This gives the t-value for the intersection

$$t = -\frac{\langle E|N\rangle}{\langle P|N\rangle - \langle E|N\rangle}$$

Substituting this t value in equation (1) gives the point

$$\begin{aligned}\langle P'| &= \langle E| - \frac{\langle E|N\rangle}{\langle P|N\rangle - \langle E|N\rangle}\left(\langle P| - \langle E|\right) \\ &= \frac{1}{\langle P|N\rangle - \langle E|N\rangle}\left(\langle P|N\rangle\langle E| - \langle E|N\rangle\langle P|\right) \\ &\sim \langle P|\left(|N\rangle\langle E| - \langle E|N\rangle I_4\right)\end{aligned}$$

The matrix describing the projection is

$$\mathcal{M} = |N\rangle\langle E| - \langle E|N\rangle I_4$$

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Substituting this t value in equation (1) gives the point

$$\begin{aligned}\langle P'| &= \langle E| - \frac{\langle E|N\rangle}{\langle P|N\rangle - \langle E|N\rangle}\left(\langle P| - \langle E|\right) \\ &= \frac{1}{\langle P|N\rangle - \langle E|N\rangle}\left(\langle P|N\rangle\langle E| - \langle E|N\rangle\langle P|\right) \\ &\sim \langle P|\left(|N\rangle\langle E| - \langle E|N\rangle I_4\right)\end{aligned}$$

The matrix describing the projection is

$$\mathcal{M} = |N\rangle\langle E| - \langle E|N\rangle I_4$$

(This is a *perspective* projection. If the eye is infinitely far away we have a *parallel* projection. How is the derivation modified in this case?)

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Curves

Curves are used in many graphical contexts. Much research has gone into developing the applications of curves during the last 30-50 years, even if the theory is often considerably older.

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Curves

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Curves can be defined in various ways

1. **parametric form**

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Curves can be defined in various ways

1. parametric form
2. **non-parametric explicit form**

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Curves can be defined in various ways

1. parametric form
2. non-parametric explicit form
3. **implicit form**

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Parameterised curves

The coordinates for a point on a curve are often given as functions of a *parameter* (it is sometimes good to think of this parameter as time). We denote the curve by $t \mapsto (x(t), y(t))$. The curve can be visualised by using the *location vector* $\mathbf{r}(t)$ pointing from the origin to the point with coordinates $(x(t), y(t))$.

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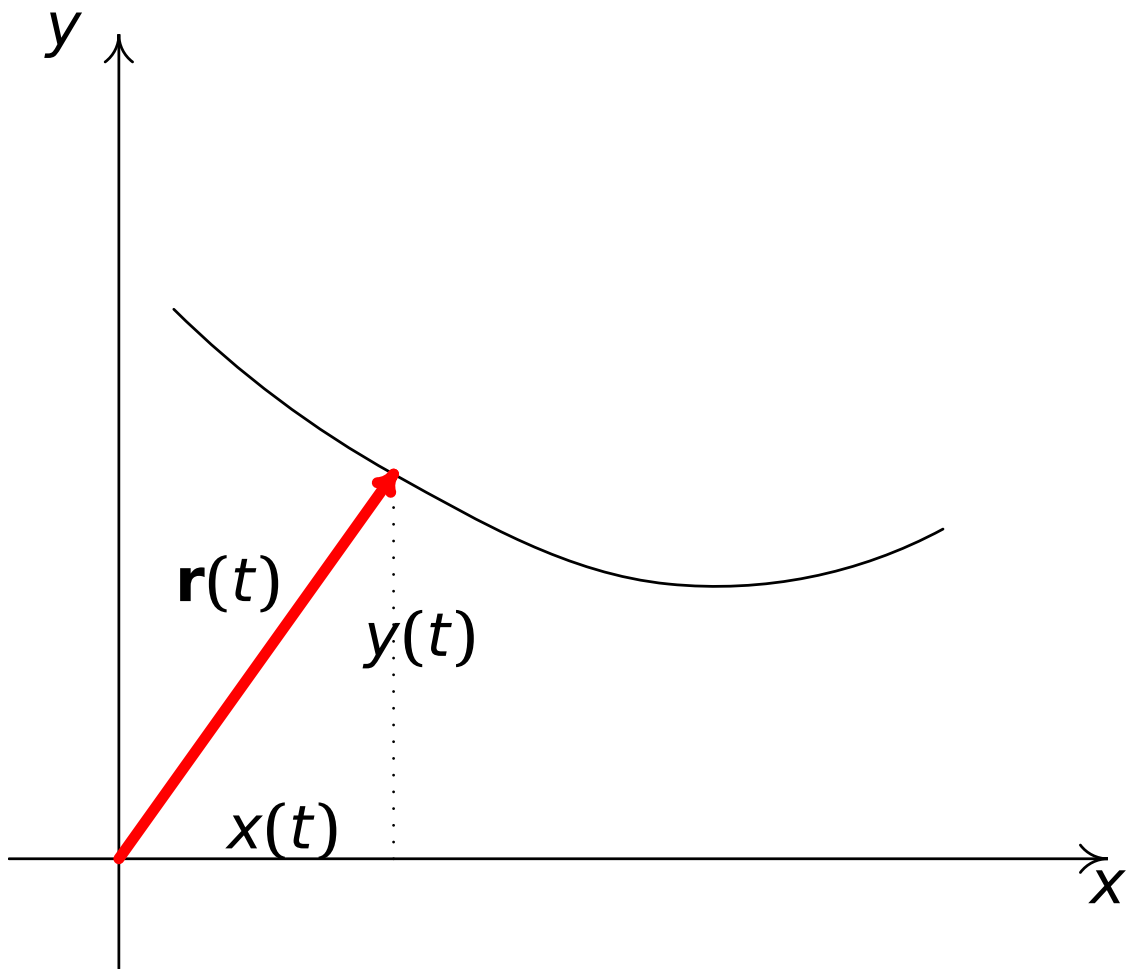

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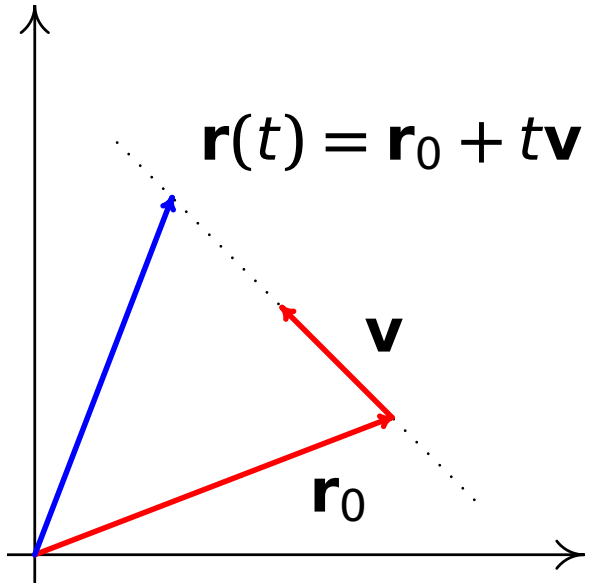
Some examples:

Straight line: $\mathbf{r}(t) = \mathbf{r} + t\mathbf{v}$



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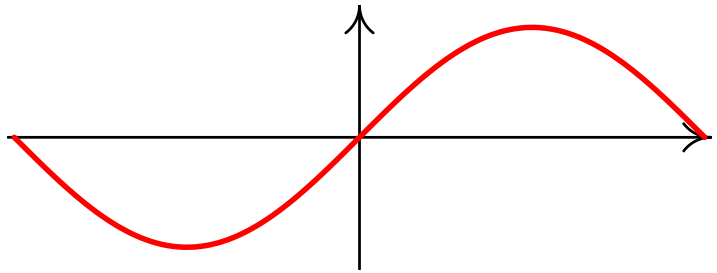
$$\mathbf{r}(x) = \begin{pmatrix} x \\ \sin x \end{pmatrix} \quad -\pi \leq x \leq \pi$$



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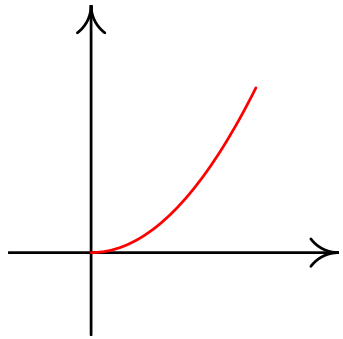
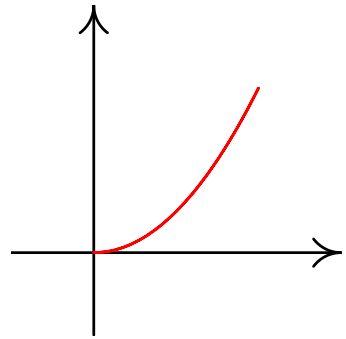
Two different parametrisation of $y = x^2$ in $0 \leq x \leq 1$





$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t \\ t^2 \end{pmatrix} \qquad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \sin t \\ \sin^2 t \end{pmatrix}$$

$$0 \leq t \leq 1 \qquad 0 \leq t \leq \pi$$

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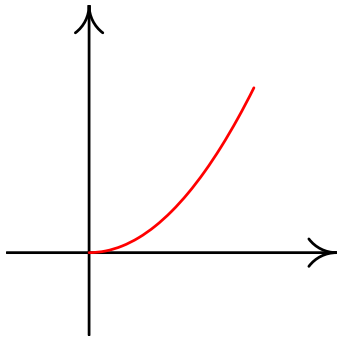
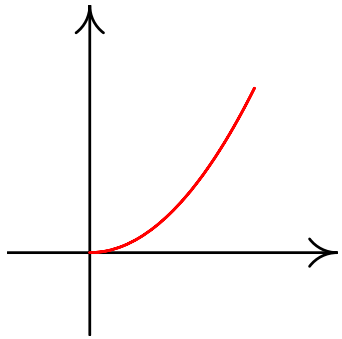
Two different parametrisation of $y = x^2$ in $0 \leq x \leq 1$





$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t \\ t^2 \end{pmatrix} \qquad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \sin t \\ \sin^2 t \end{pmatrix}$$

$$0 \leq t \leq 1 \qquad 0 \leq t \leq \pi$$

What is the difference between the curves?

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Parametric curves are *oriented*, that is they have a *starting point*, and an *endpoint* which together give the *orientation*.

A parametric curve is an excellent tool to describe a moving particle in physics. We let the position of the particle be given by $(x(t), y(t))$, where t is the time. This is easily generalised to higher dimensions.

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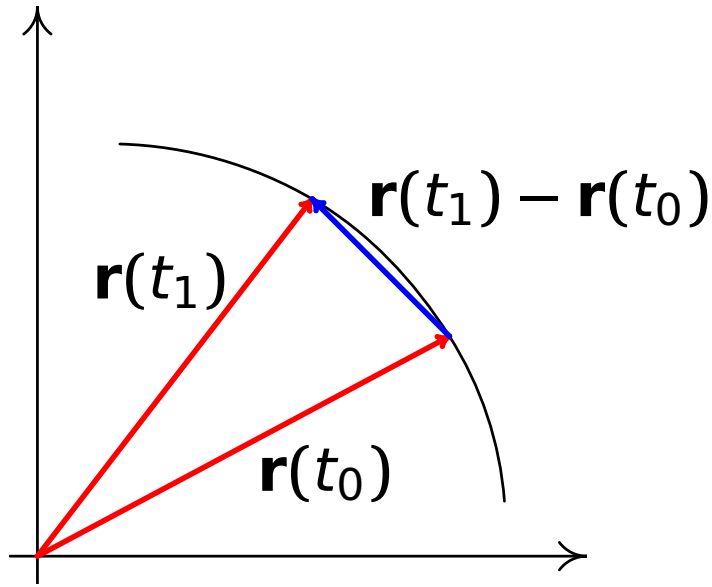
Let a particle move along a curve $\mathbf{r}(t) = (x(t), y(t))$.



If we let $t_1 \to t_0$ we obtain in a natural way the *derivative* which in this case is a vector itself

$$\mathbf{r}'(t_0) = \lim_{t_1 \to t_0} \frac{\mathbf{r}(t_1) - \mathbf{r}(t_0)}{t_1 - t_0} = \begin{pmatrix} x'(t_0) \\ y'(t_0) \end{pmatrix}$$

This vector has a geometric interpretation as the *tangent vector* to the graph, and a physical interpretation (if t is time) as the *velocity vector*

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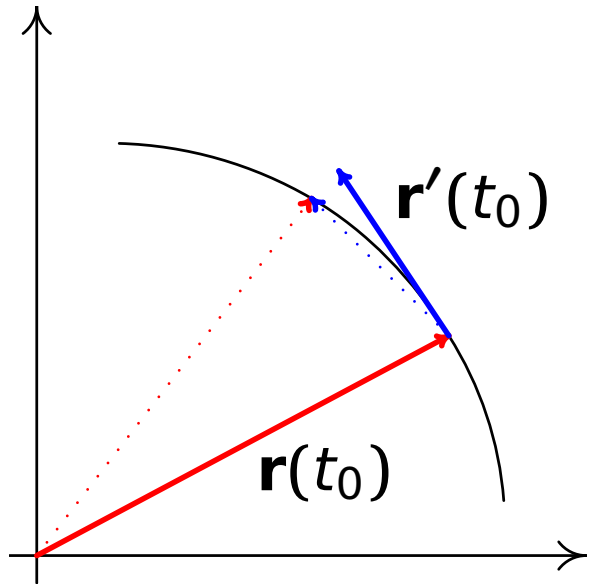

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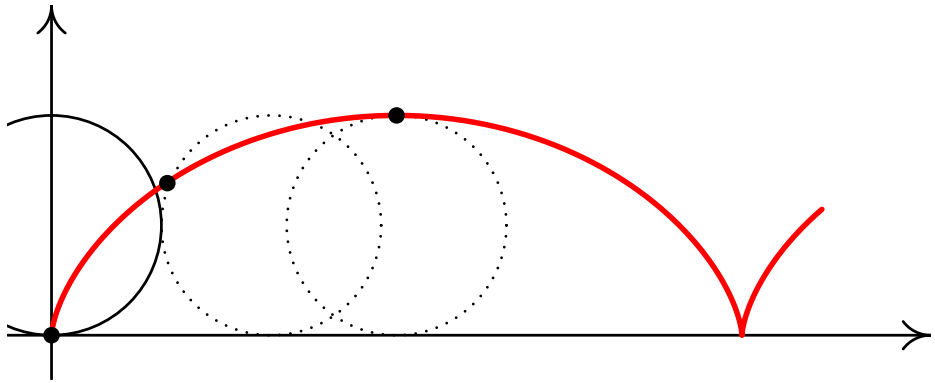
A *cycloid* is the curve that is generated by a point on the perimeter of a circle rolling on a line. The equations are given by

$$x = R(t - \sin t)$$
$$y = R(1 - \cos t)$$



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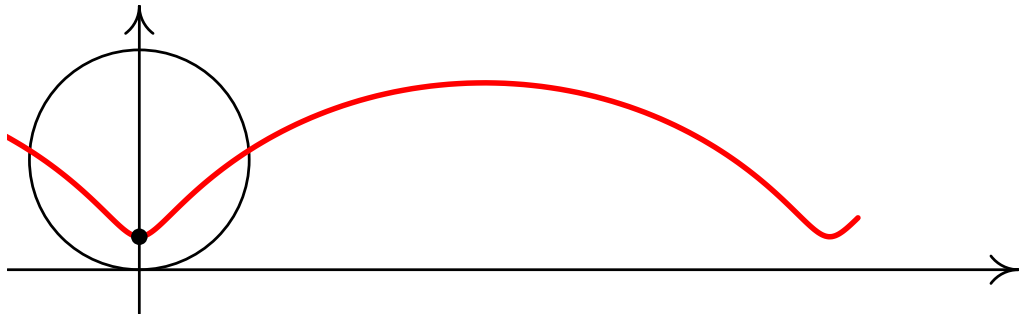
A *shortened cycloid* is generated by a point inside the rolling circle. The equations are ($c < R$)

$$x = Rt - c\sin t$$
$$y = R - c\cos t$$



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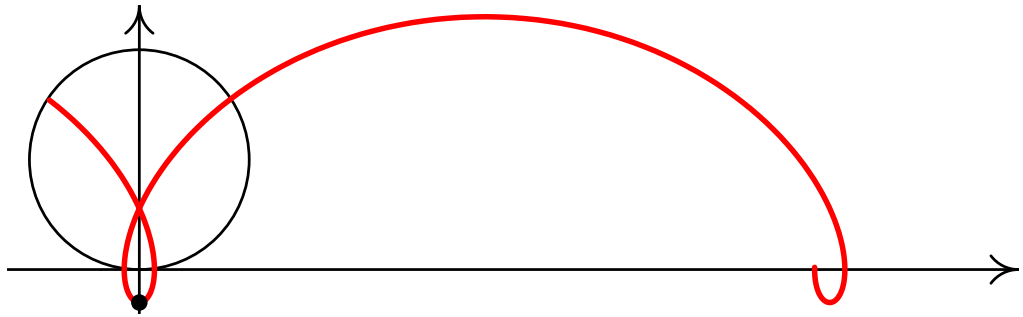
An *extended* cycloid is generated by a point outside the rolling circle (give an example of how this can be realised). The equations are ($c > R$)

$$x = Rt - c\sin t$$
$$y = R - c\cos t$$



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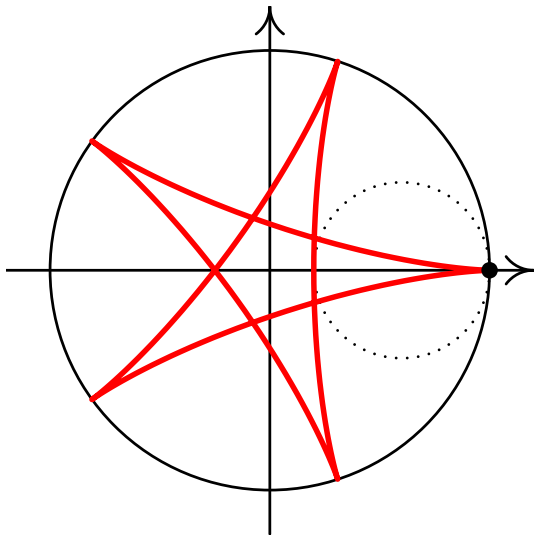
A *hypocycloid* is generated by a point on the perimeter of a circle (radius r) rolling inside another circle (radius R). The equations are

$$x = (R-r)\cos t + r\cos(1-R/r)t$$
$$y = (R-r)\sin t + r\sin(1-R/r)t$$



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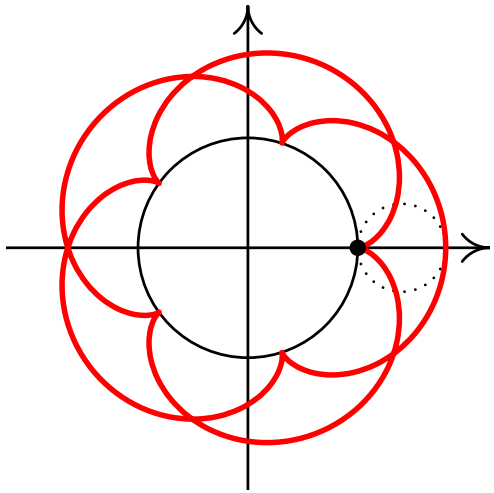
An *epicycloid* is generated by a point on the perimeter of a circle (radius r) rolling outside another circle (radius R). The equations are

$$x = (R + r)\cos t - r\cos(1 + R/r)t$$
$$y = (R + r)\sin t - r\sin(1 + R/r)t$$



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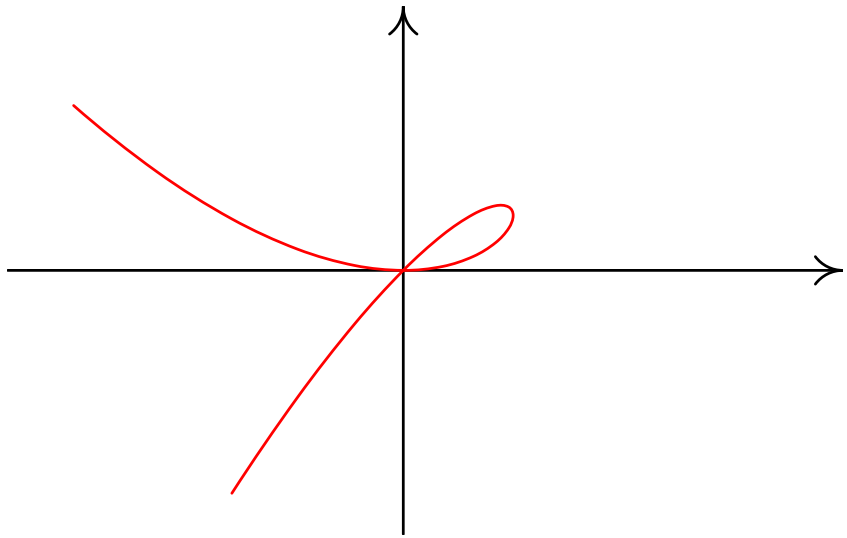
Some ”polynomial curves”

$$x = t(1-t)$$
$$y = t^2(1-t)$$



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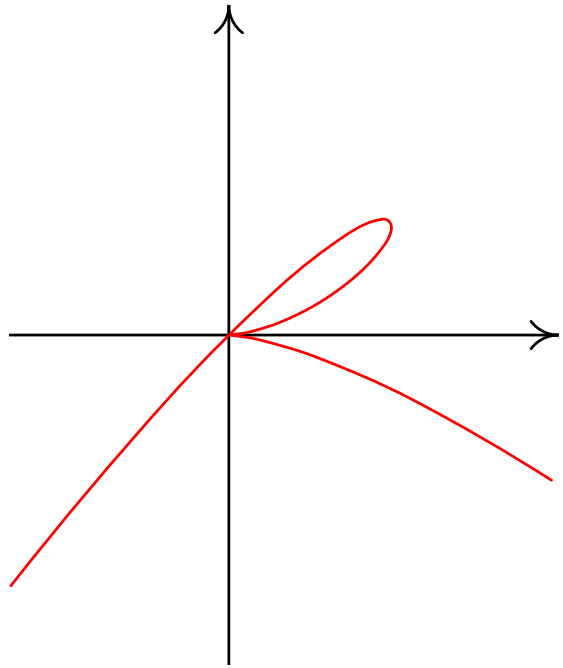
$$x = t^2(1-t)$$
$$y = t^3(1-t)$$



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Different techniques of constructing curves

interpolation Here we construct curves *going through* a number of given points. Examples: Lagrange interpolation, Newton interpolation, Hermite interpolation, various types of splines . . .

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Different techniques of constructing curves

interpolation Here we construct curves *going through* a number of given points. Examples: Lagrange interpolation, Newton interpolation, Hermite interpolation, various types of splines . . .

approximation Here we construct curves *determined by* a number of given points, but *not necessarily going through all* of the points. Examples: Bezier curves, B-splines . . .

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The Lagrange polynomial

Given $n+1$ data points $\mathbf{P}_0$, $\mathbf{P}_1$, $\ldots$ $\mathbf{P}_n$, we want to find a function $\mathbf{P}(t)$ which passes through all the points $\mathbf{P}_0, \ldots \mathbf{P}_n$, such that $\mathbf{P}(t_0) = \mathbf{P}_0$, $\mathbf{P}(t_1) = \mathbf{P}_1, \ldots$, $\mathbf{P}(t_n) = \mathbf{P}_n$. Here we have $t_0 = 0$, $t_n = 1$ and $t_1, \ldots t_{n-1}$ are certain values between 0 and 1 (called *knot* values).

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The Lagrange polynomial is written as

$$\mathbf{P}(t) = \sum_{k=0}^{n} \mathbf{P}_k L_k^n(t)$$

This is a weighted sum of the given points, where the weights (or basis functions) are given by

$$L_k^n(t) = \frac{\prod_{j \neq k}(t - t_j)}{\prod_{j \neq k}(t_k - t_j)} \quad \left(= \begin{cases} 1 & t = t_k \\ 0 & t = t_j,\ j \neq k \end{cases} \right)$$

Note that $\sum_{k=0}^{n} L_k^n(t) \equiv 1$. (Prove this!)

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The Newton polynomial

We have again $n+1$ data points $\mathbf{P}_0$, $\mathbf{P}_1$, $\ldots \mathbf{P}_n$, and assigned knot values

$$t_0 = 0 < t_1 < \cdots < t_n = 1.$$

We are looking for a curve of the form

$$\mathbf{P}(t) = \sum_{i=1}^{n} N_i(t)\mathbf{A}_i$$

where the basis functions depend on the knot values only, and the (yet unknown) coefficients $\mathbf{A}_i$ depend on the data points.

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The basis functions are given by

$$N_0(t) = 1$$
$$N_1(t) = t - t_0$$
$$\vdots$$
$$N_i(t) = (t - t_0)(t - t_1) \cdots (t - t_{i-1}),\ i = 2, \ldots, n$$

Note that each basis function N_i depends only on the previous knot values.

To calculate the coefficients we get the system of equations

$$\mathbf{P}_0 = \mathbf{P}(t_0) = \mathbf{A}_0$$
$$\mathbf{P}_1 = \mathbf{P}(t_1) = \mathbf{A}_0 + \mathbf{A}_1(t_1 - t_0)$$
$$\mathbf{P}_2 = \mathbf{P}(t_2) = \mathbf{A}_0 + \mathbf{A}_1(t_2 - t_0) + \mathbf{A}_2(t_2 - t_0)(t_2 - t_1)$$
$$\vdots$$

These can be solved recursively and the solution is

$$\mathbf{A}_0 = \mathbf{P}_0$$

$$\mathbf{A}_1 = \frac{\mathbf{P}_1 - \mathbf{P}_0}{t_1 - t_0}$$

$$\mathbf{A}_2 = \frac{\mathbf{P}_2 - \mathbf{P}_0 - \frac{\mathbf{P}_1 - \mathbf{P}_0}{t_1 - t_0}}{(t_2 - t_0)(t_2 - t_1)} = \frac{\frac{\mathbf{P}_2 - \mathbf{P}_1}{t_2 - t_1} - \frac{\mathbf{P}_1 - \mathbf{P}_0}{t_1 - t_0}}{t_2 - t_0}$$

$$\ldots$$

This becomes very complicated quickly and Newton introduced the concept of *divided differences* to simplify. The divided difference of the knots $t_i t_k$ is defined as

$$[t_i t_k] = \frac{\mathbf{P}_i - \mathbf{P}_k}{t_i - t_k}$$

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The solution can now be written

$$\mathbf{A}_0 = \mathbf{P}_0$$

$$\mathbf{A}_1 = \frac{\mathbf{P}_1 - \mathbf{P}_0}{t_1 - t_0} = [t_1 t_0]$$

$$\mathbf{A}_2 = [t_2 t_1 t_0] = \frac{[t_2 t_1] - [t_1 t_0]}{t_2 - t_0}$$

$$\mathbf{A}_3 = [t_3 t_2 t_1 t_0] = \frac{[t_3 t_2 t_1] - [t_2 t_1 t_0]}{t_3 - t_0}$$

$$\vdots$$

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Hermite interpolation

This method gives a parametric cubic curve (degree 3) connecting two given points. Such a curve has 4 coefficients and is not uniquely determined by two points. The Hermite method requires that the tangent vectors at the start and end points are also given.

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Let the curve be given by $\mathbf{P}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$ for $0 \leq t \leq 1$. The tangent vector to the curve is of course the derivative

$$\frac{d\mathbf{P}(t)}{dt} = \mathbf{P}'(t) = 3\mathbf{a}t^2 + 2\mathbf{b}t + \mathbf{c}$$

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Let the given data points be $\mathbf{P}_0$, $\mathbf{P}_1$ with tangent vectors $\mathbf{P}_0^t$, $\mathbf{P}_1^t$ respectively. With knot values 0 and 1 we get the system

$$\begin{aligned}
\mathbf{a}\cdot 0^3 + \mathbf{b}\cdot 0^2 + \mathbf{c}\cdot 0 + \mathbf{d} &= \mathbf{P}_0 \\
\mathbf{a}\cdot 1^3 + \mathbf{b}\cdot 1^2 + \mathbf{c}\cdot 1 + \mathbf{d} &= \mathbf{P}_1 \\
3\mathbf{a}\cdot 0^2 + 2\mathbf{b}\cdot 0 + \mathbf{c} &= \mathbf{P}_0^t \\
3\mathbf{a}\cdot 1^2 + 2\mathbf{b}\cdot 1 + \mathbf{c} &= \mathbf{P}_1^t
\end{aligned}$$

This is easily solved and gives

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$$\mathbf{a} = 2\mathbf{P}_0 - 2\mathbf{P}_1 + \mathbf{P}_0^t + \mathbf{P}_1^t$$
$$\mathbf{b} = -3\mathbf{P}_0 + 3\mathbf{P}_1 - 2\mathbf{P}_0^t - \mathbf{P}_1^t$$
$$\mathbf{c} = \mathbf{P}_0^t$$
$$\mathbf{d} = \mathbf{P}_0$$

The function $\mathbf{P}$ is then

$$\mathbf{P}(t) = (2\mathbf{P}_0 - 2\mathbf{P}_1 + \mathbf{P}_0^t + \mathbf{P}_1^t)t^3 + (-3\mathbf{P}_0 + 3\mathbf{P}_1 - 2\mathbf{P}_0^t - \mathbf{P}_1^t)t^2 + \mathbf{P}_0^t t + \mathbf{P}_0$$

We can rearrange this as

$$\begin{aligned}\mathbf{P}(t) &= (2t^3 - 3t^2 + 1)\mathbf{P}_0 + (-2t^3 + 3t^2)\mathbf{P}_1 \\ &\quad + (t^3 - 2t^2 + t)\mathbf{P}_0^t + (t^3 - t^2)\mathbf{P}_1^t \\ &= F_1(t)\mathbf{P}_0 + F_2(t)\mathbf{P}_1 + F_3(t)\mathbf{P}_0^t + F_4(t)\mathbf{P}_1^t \\ &= (F_1(t), F_2(t), F_3(t), F_4(t))(\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_0^t, \mathbf{P}_1^t)^T \\ &= \mathbf{F}(t)\mathbf{B}\end{aligned}$$

$\mathbf{B}$ is the column $(\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_0^t, \mathbf{P}_1^t)^T$ and $\mathbf{F}(t)$ is the row $(F_1(t), F_2(t), F_3(t), F_4(t))$. The functions $F_i(t)$ are called the *Hermite blending functions*.

We can write this in matrix form

$$\mathbf{F}(t) = \begin{pmatrix} t^3 & t^2 & t & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \mathbf{T}(t)\mathbf{H}$$

which gives

$$\mathbf{P}(t) = \mathbf{T}(t)\mathbf{H}\mathbf{B}$$

$$= \begin{pmatrix} t^3 & t^2 & t & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_0^t \\ \mathbf{P}_1^t \end{pmatrix}$$

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Some problems with interpolation:

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Some problems with interpolation:

1. **high-degree polynomials**

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Some problems with interpolation:

1. high-degree polynomials
2. **oscillations**

Some problems with interpolation:

1. high-degree polynomials
2. oscillations
3. **unique**

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Some problems with interpolation:

1. high-degree polynomials
2. oscillations
3. unique

How can we alleviate these problems?

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Bézier curves

Paul Bézier while working for Rénault in the early 1960's.
Paul Faget de Casteljau while working for Citroën in 1959 (never published)

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Bézier curves

Paul Bézier while working for Rénault in the early 1960's.
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Uses *control points* and produces an approximating curve. The curve passes through the first and last of the points, but is "pulled" toward the others, with the strongest pull when it is nearest the point.

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We describe two methods to construct the Bézier curve: a weighted sum and a linear interpolation.

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Bernstein-polynomials

Using the binomial theorem we can expand

$$1 = ((1-t)+t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k}$$

Define the Bernstein-polynomials by

$$B_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}$$

Note that we have $B_{n,k}(t) \geq 0$ for $t \in [0,1]$ and that

$$\sum_{k=0}^{n} B_{n,k}(t) = 1.$$

We now define the Bézier curve with control points $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n$ by

$$\mathbf{P}(t) = \sum_{k=0}^{n} B_{n,k}(t)\mathbf{P}_k = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} \mathbf{P}_k$$

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We see easily that the curve passes the points $\mathbf{P}_0$ (for $t = 0$) and $\mathbf{P}_n$ (for $t = 1$). The other control points ”pull” the curve and *all* control points influence the shape of the curve.

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2 control points give a straight line:

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With 4 (or more) control points interesting things begin to happen

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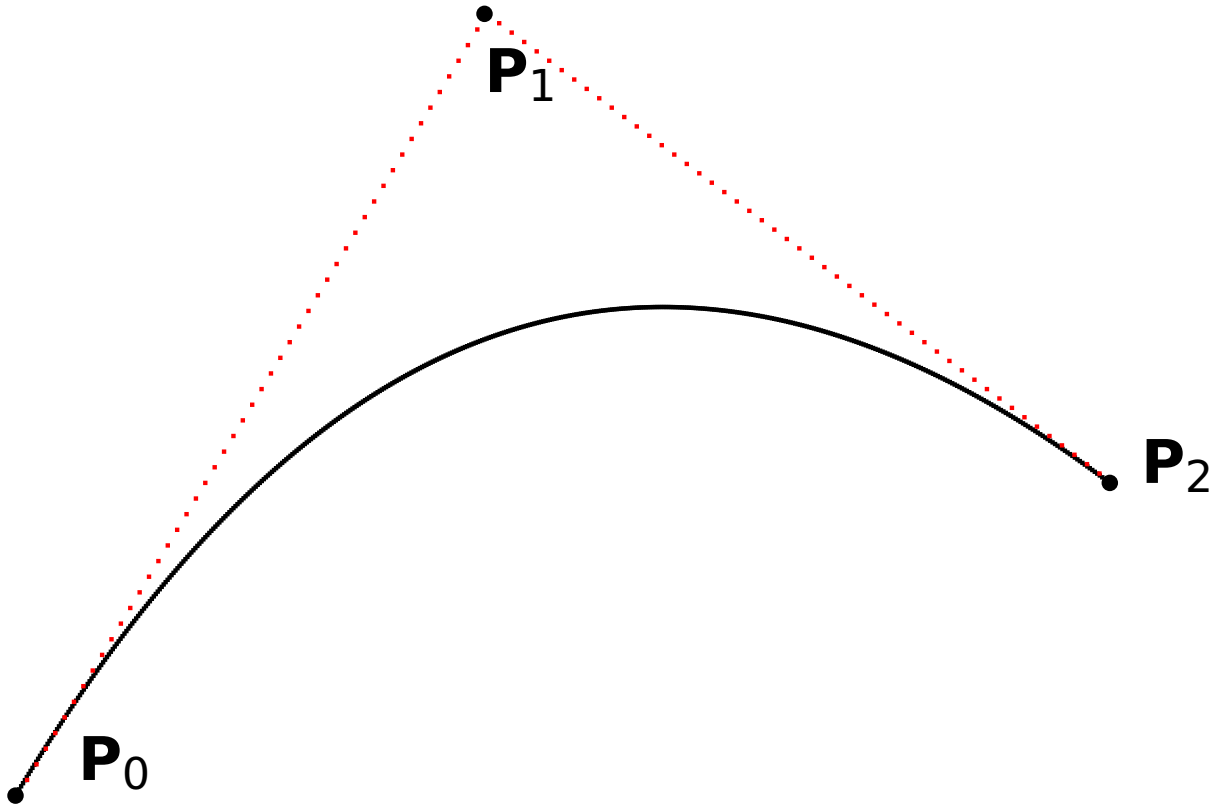

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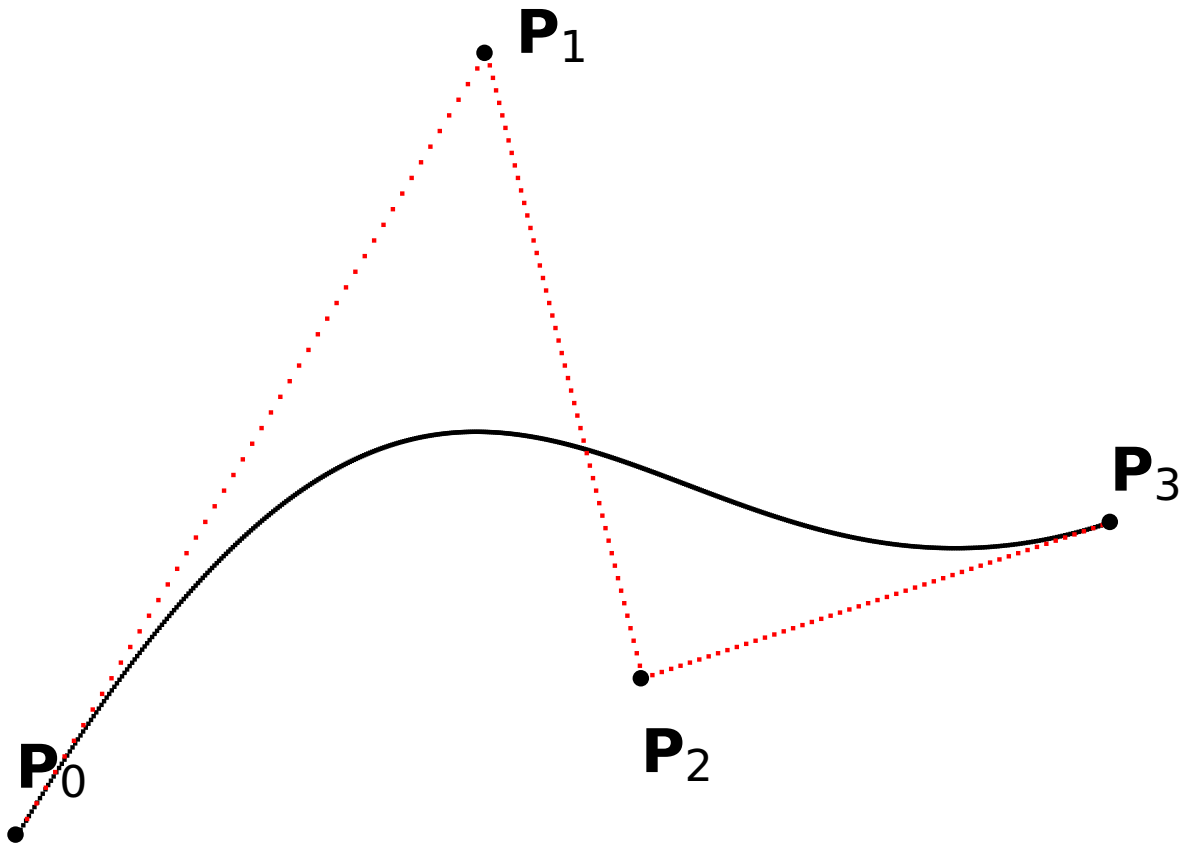

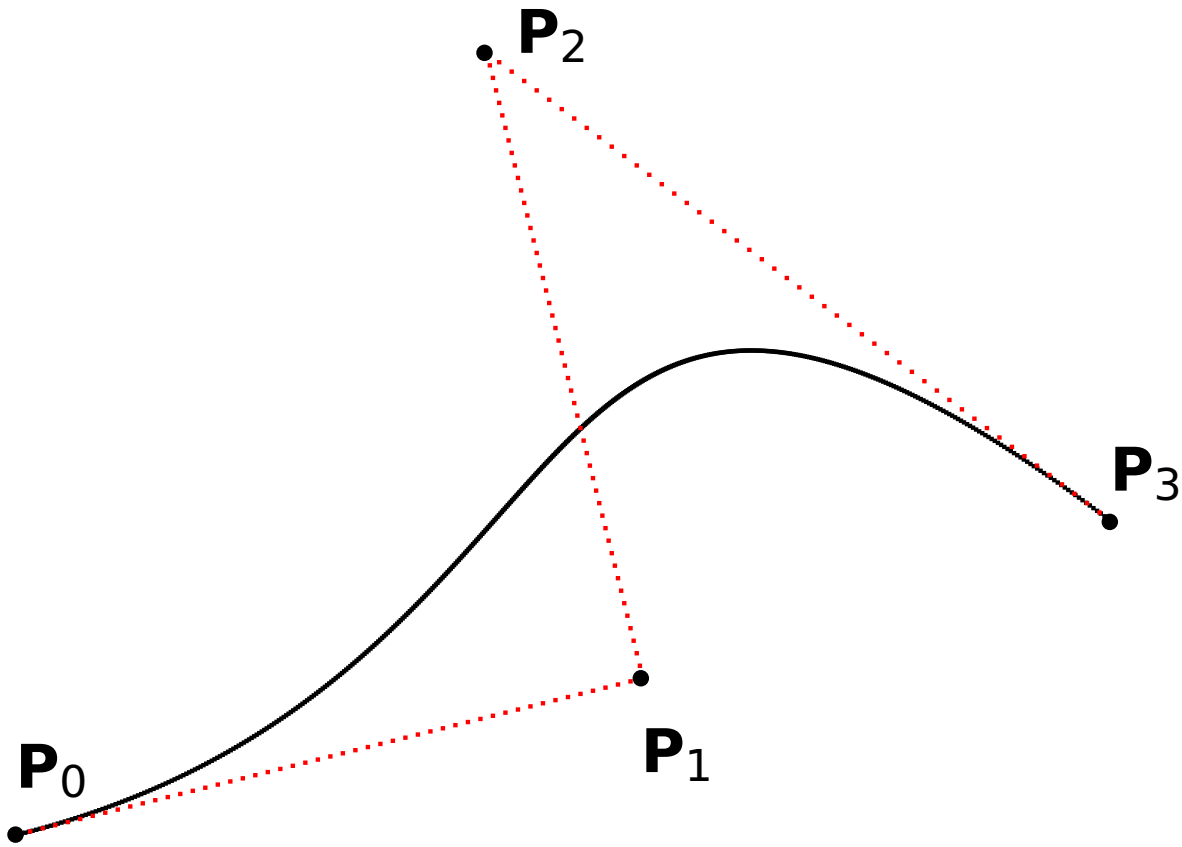
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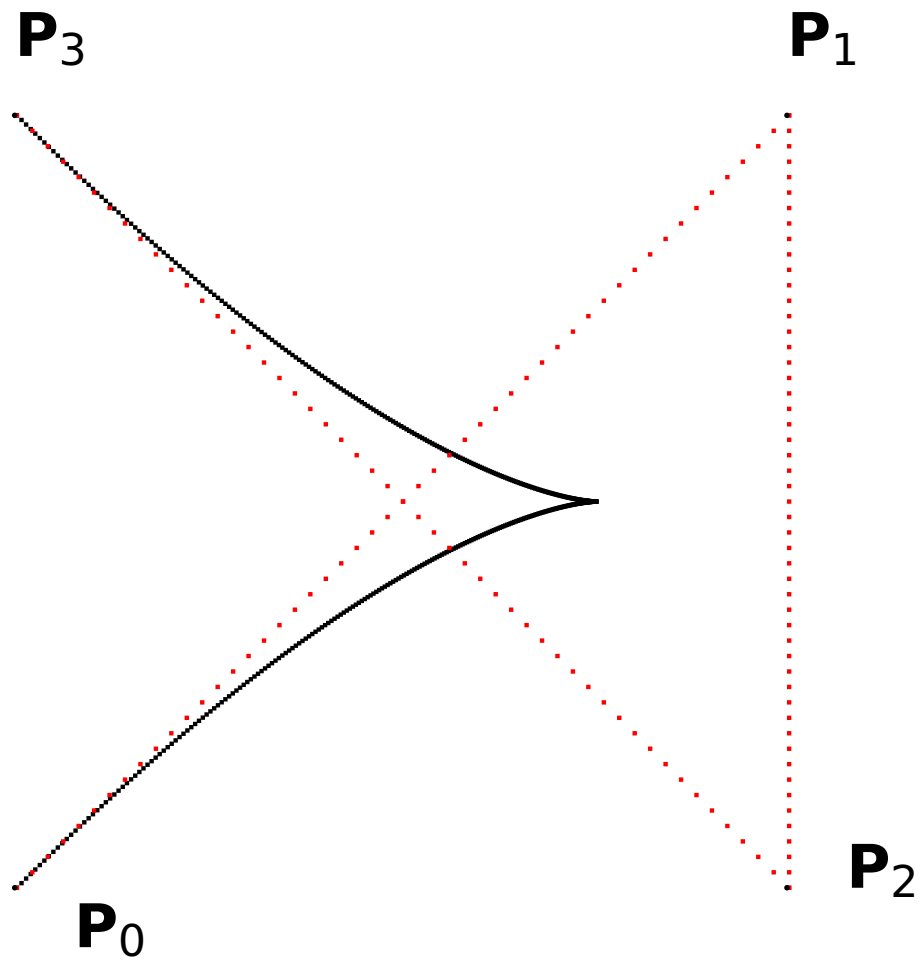

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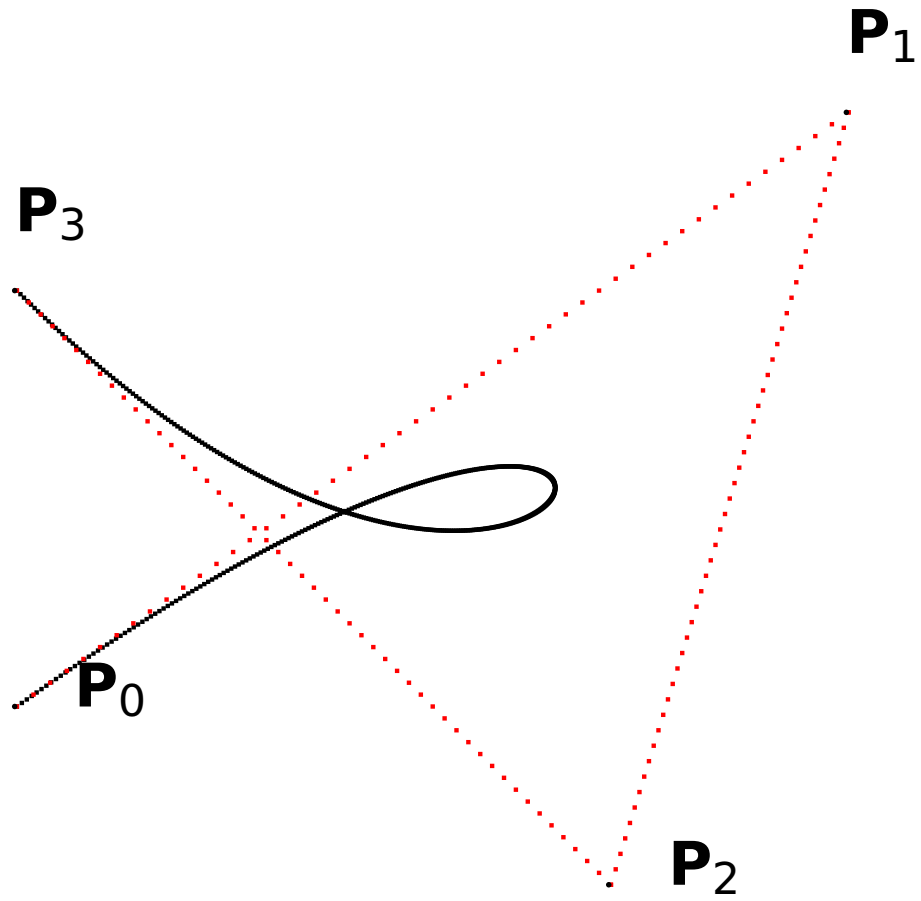

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Some properties of Bézier curves

1. goes through the first and the last of the control points.

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Some properties of Bézier curves

1. goes through the first and the last of the control points.
2. The tangent directions at the first and last points are determined by the second and second last points, respectively.

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Some properties of Bézier curves

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3. local properties of the curve are ”smoothed out”

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Some properties of Bézier curves

1. goes through the first and the last of the control points.
2. The tangent directions at the first and last points are determined by the second and second last points, respectively.
3. local properties of the curve are ”smoothed out”
4. the curve lies in the convex hull of the control points

Definition: The *convex hull* of the points $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n$ is the set of all points of the form

$$\lambda_0\mathbf{P}_0 + \lambda_1\mathbf{P}_1 + \cdots + \lambda_n\mathbf{P}_n$$

where the λ_i satisfy

$$\lambda_0 + \lambda_1 + \cdots + \lambda_n = 1$$

and

$$0 \leq \lambda_i \leq 1,\ i = 0, 1, \ldots, n$$

(Compare with the properties of the Bernstein-polynomials)

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The Casteljau construction

We illustrate the other method of deriving a Bézier curve by an example with 4 control points $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$.

Construct 3 new points as weighted means

$$\mathbf{U}_0 = (1-t)\mathbf{P}_0 + t\mathbf{P}_1$$
$$\mathbf{U}_1 = (1-t)\mathbf{P}_1 + t\mathbf{P}_2$$
$$\mathbf{U}_2 = (1-t)\mathbf{P}_2 + t\mathbf{P}_3$$

From this 3 points we now construct 2 new points, again as weighted means

$$\mathbf{V}_0 = (1-t)\mathbf{U}_0 + t\mathbf{U}_1$$
$$\mathbf{V}_1 = (1-t)\mathbf{U}_1 + t\mathbf{U}_2$$

and finally,

$$\mathbf{P}(t) = \mathbf{W} = (1-t)\mathbf{V}_0 + t\mathbf{V}_1$$

If we substitute back we find that

$$\mathbf{P}(t) = (1-t)^3\mathbf{P}_0 + 3t(1-t)^2\mathbf{P}_1 + 3t^2(1-t)\mathbf{P}_2 + t^3\mathbf{P}_3$$

which obviously is the Bézier curve.

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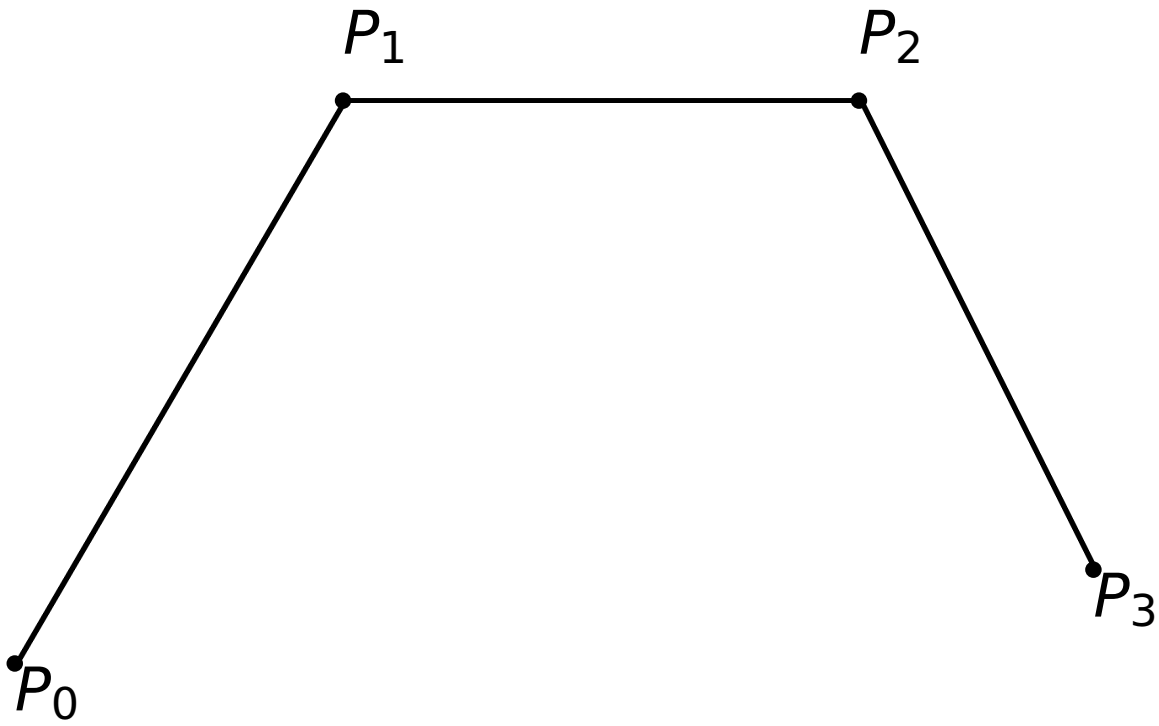
The whole procedure can be illustrated geometrically



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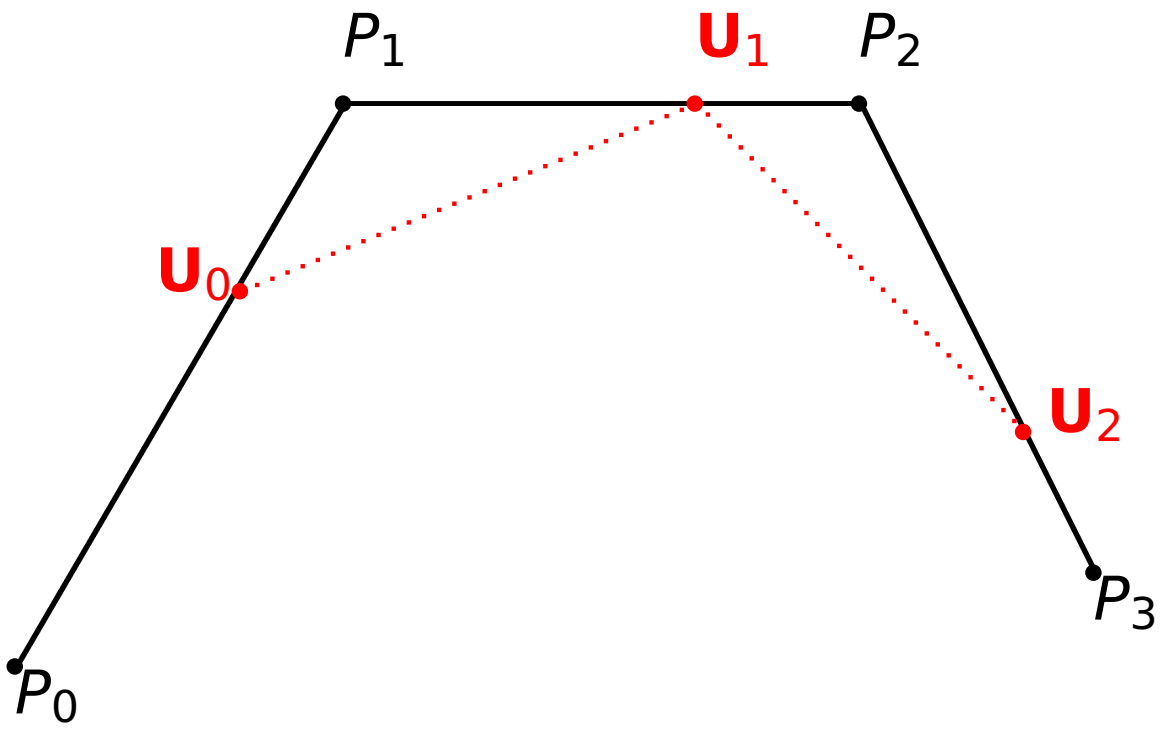

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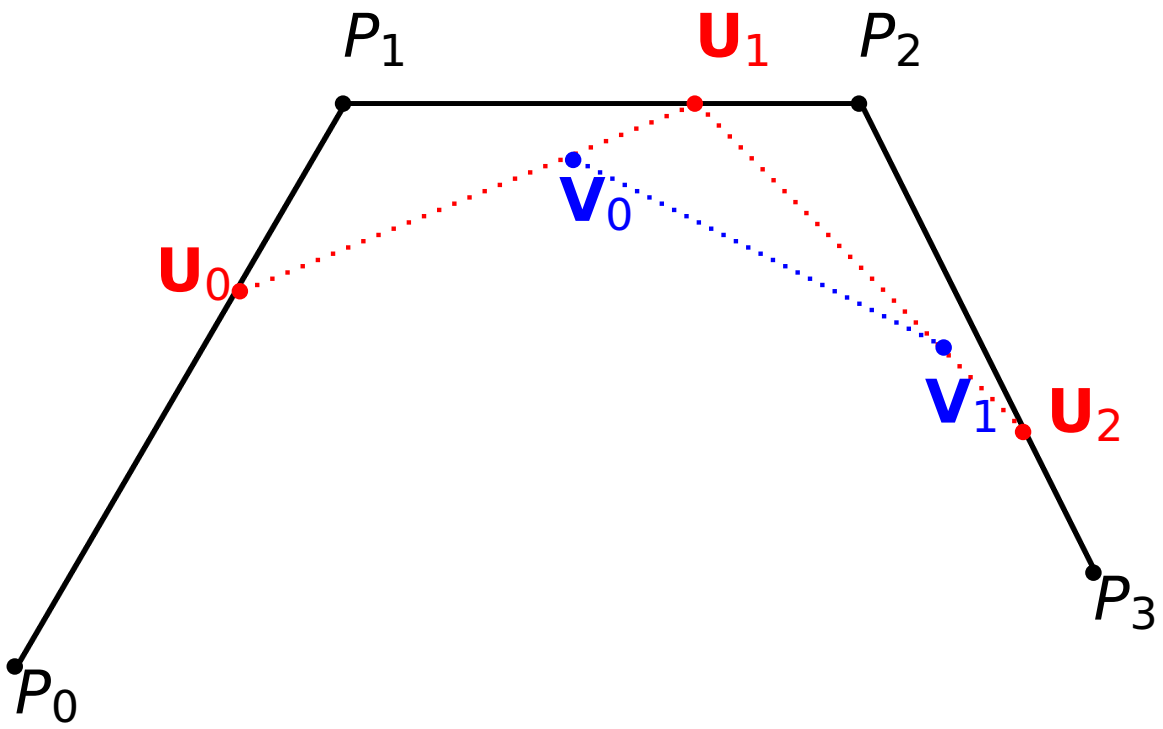

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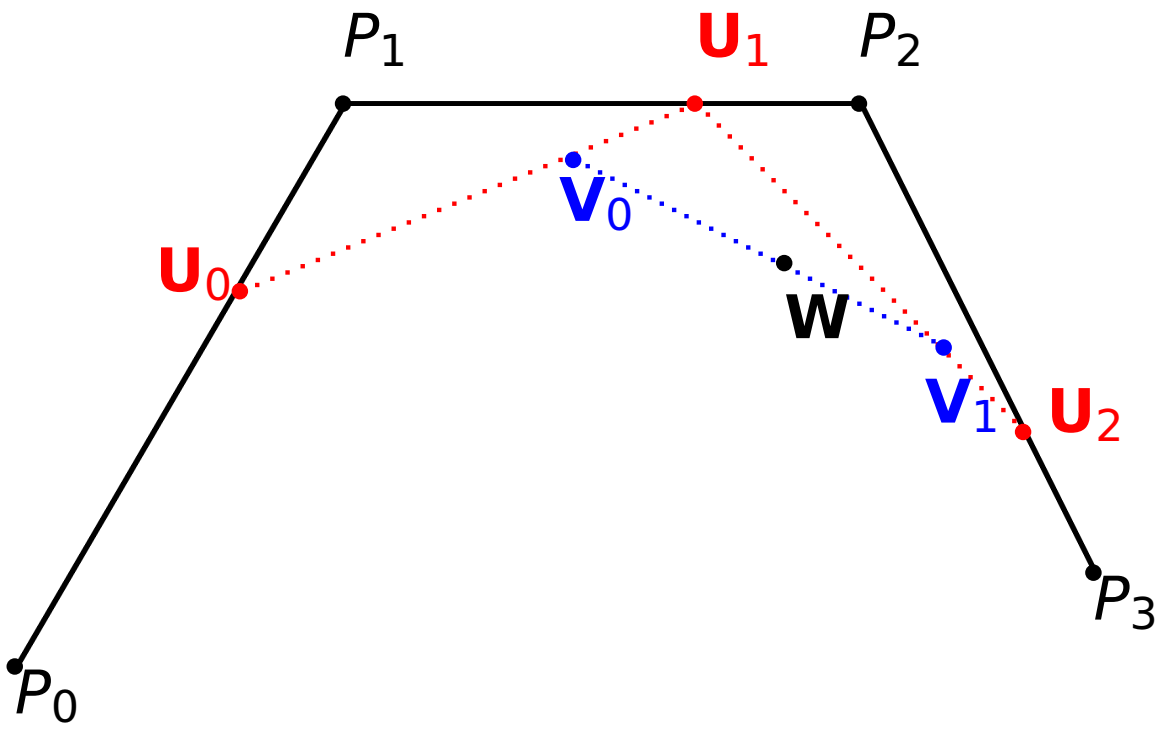

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If we put in the resulting Bézier curve it looks like