Proof. Let $u = \sum_{i=i}^{m} u_i$, u_i in N_i and $v = \sum_{i=1}^{n} v_i$, v_i in N_i'. Then

$$\|u\|^2 = \sum_{i,j} \langle u_i, u_j \rangle = \sum \|u_i\|^2 + \varepsilon' \quad \text{where} \quad \varepsilon' = \sum_{i \neq j} \langle u_i, u_j \rangle$$

$$|\varepsilon'| \leq \varepsilon \sum_{i \neq j} \|u_i\| \cdot \|u_j\| = \varepsilon \left[\Big(\sum \|u_i\| \Big) \Big(\sum \|u_j\| \Big) - \sum \|u_i\|^2 \right]$$
$$\leq \varepsilon(m-1) \sum \|u_i\|^2.$$

Thus

$$\|u\|^2 \geq \Big[1 - (m-1)\varepsilon \Big] \sum \|u_i\|^2 \geq \frac{1}{2} \sum \|u_i\|^2$$

if $2m\varepsilon < 1$ and similarly for v. Then

$$|\langle u, v \rangle| = \Big| \sum_i \sum_j \langle u_i, v_j \rangle \Big| \leq \varepsilon \sum_i \sum_j \|u_i\| \cdot \|v_j\| = \varepsilon \Big(\sum \|u_i\| \Big) \Big(\sum \|v_j\| \Big)$$
$$\leq \varepsilon \cdot m \cdot n \Big[\sum \|u_i\|^2 \Big]^{1/2} \cdot \Big[\sum \|v_j\|^2 \Big]^{1/2} \leq 2mn\varepsilon \cdot \|u\| \cdot \|v\|.$$

and thus u and v are $2mn\varepsilon$-orthogonal. If $2m\varepsilon \geq 1$ or $2n\varepsilon \geq 1$, then $2mn\varepsilon \geq 1$ and the statement is trivially satisfied. □

I.1.3 Some Auxiliary Theorems on Approxmation of Subspaces

Lemma I.1:23. $N_1 \subset_\varepsilon N_2$ if for every vector u_1 in N_1 there exists a u_2 in N_2 with

$$\|u_1 - u_2\| \leq \varepsilon \cdot \|u_1\|$$

Proof. A direct consequence of lemma I.1:4 and remark 1 after definition I.1.1:3 if N_1 and N_2 are closed. If N_1 or N_2 are not closed, let M_1 and M_2 be the closures of N_1 and N_2. The statement for N_1 and N_2 then follows from the statement for M_1 and M_2 by using remark 3 after theorem I.1:8. □

Lemma I.1:24. $N_1 \subset_\varepsilon M_2$ iff for every u in N_1

$$\|u - P_2 u\| \leq \varepsilon \|u\|,$$

where P_2 is the projection on M_2

Proof. A direct consequence of lemma I.1:4 and remark 1 after definition I.1:3 □

Lemma I.1:25. If $N_1 \subset_\varepsilon M_2$ and P_2 is the projection on M_2 then

$$N_1 \subset_\varepsilon P_2 N_1 \quad \text{and}$$
$$P_2 N_1 \subset_\varepsilon N_1.$$

Proof. The first statement follows directly from lemma I.1:24.

Let u be an arbitrary nonzero vector in $P_2 N_1$. Then u is of the form $u = P_2 v$ with v in N_1 and lemma I.1:4 gives $\text{dist}(u, v) = \text{dist}(v, u) = \text{dist}(v, P_2 v) = \text{dist}(v, M_2) \leq$

ε. Thus for arbitrary nonzero u in $P_2 N_1$ there is a v in N_1 with $\text{dist}(u,v) \leq \varepsilon$. Hence $\text{dist}(P_2 N_1, N_1) \leq \varepsilon$ and $P_2 N_1 \subset_\varepsilon N_1$. □

Lemma I.1:26. Suppose that M_1 and M_2 are orthogonal and that $M_1 \subset_{\varepsilon_1} M_1'$ and $M_2 \subset_{\varepsilon_2} M_2'$. Then $M_1 \oplus M_2 \subset_{\varepsilon_1+\varepsilon_2} M_1' \oplus M_2'$.

Proof. Let u be an arbitrary vector in $M_1 \oplus M_2$. Then u is of the form $u = u_1 + u_2$ with u_1 and u_2 in M_1 and M_2 respectively. By lemma I.1:23 there exists u_1' and u_2' in M_1' and M_2' respectively with $\|u_1 - u_1'\| \leq \varepsilon_1 \|u_1\|$ and $\|u_2 - u_2'\| \leq \varepsilon_2 \|u_2\|$. Hence there exists an $u' = u_1' + u_2'$ in $M_1' \oplus M_2'$ with

$$\begin{aligned}\|u - u'\| &\leq \|u_1 - u_1'\| + \|u_2 - u_2'\| \leq \varepsilon_1 \|u_1\| + \varepsilon_2 \|u_2\| \\ &\leq \left(\varepsilon_1^2 + \varepsilon_2^2\right)^{1/2} \left(\|u_1\|^2 + \|u_2\|^2\right)^{1/2} = \left(\varepsilon_1^2 + \varepsilon_2^2\right)^{1/2} \|u\| \\ &\leq \left(\varepsilon_1 + \varepsilon_2\right) \|u\|\end{aligned}$$

since u_1 and u_2 are orthogonal and thus $M_1 \oplus M_2 \subset_{\varepsilon_1+\varepsilon_2} M_1' \oplus M_2'$ □

Lemma I.1:27. Suppose $M \subset_{\varepsilon'} M'$ and $M \subset_{\varepsilon''} M''$. Then $M \subset_{\varepsilon'+\varepsilon''} P'' M'$ where P'' is the projection on M''.

Proof. Let u be an arbitrary vector in M and let P' be the projection on M'. Then by lemma I.1:24 $\|P'u - u\| \leq \varepsilon' \|u\|$, $\|P''u - u\| \leq \varepsilon'' \|u\|$ and

$$\begin{aligned}\|P''P'u - u\| &\leq \|P''P'u - P''u\| + \|P''u - u\| \leq \|P''(P'u - u)\| + \varepsilon'' \|u\| \\ &\leq \|P'u - u\| + \varepsilon'' \|u\| \leq \left(\varepsilon' + \varepsilon''\right) \|u\|.\end{aligned}$$

Sinse $P''P'u$ is in $P''M'$, lemma I.1:23 then gives that $M \subset_{\varepsilon'+\varepsilon''} P''M'$. □

Lemma I.1:28. Suppose $M \subset_{\varepsilon'} M'$ and $M \subset_{\varepsilon''} M''$. Then $P''M \subset_{\varepsilon'+\varepsilon''} M'$ where P'' is the projection on M''.

Proof. Let u be a vector in M. Then we have that $\text{dist}(P''u, u) = \text{dist}(u, P''u) \leq \varepsilon''$, $\text{dist}(u, P'u) \leq \varepsilon'$ and by lemma I.1:5

$$\text{dist}(P''u, P'u) \leq \text{dist}(P''u, u) + \text{dist}(u, P'u) \leq \varepsilon' + \varepsilon''.$$

Since $P'u$ is in M', this shows that $\text{dist}(P''u, M') \leq \varepsilon' + \varepsilon''$ and since an arbitrary vector in $P''M$ is of the form $P''u$ with u in M, this also shows that $\text{dist}(P''M, M') \leq \varepsilon' + \varepsilon''$ and it follows that $P''M \subset_{\varepsilon'+\varepsilon''} M'$. □

Lemma I.1:29. Suppose that $M_1 = R(P_1)$, $M_2 = R(P_2)$, P commutes with P_1 and P_2 and $M_1 \subset_\varepsilon M_2$. Then $PM_1 \subset_\varepsilon PM_2$.

Proof. For arbitrary u in PM_1, we have, since $PM_1 \subset M_1$, by lemma I.1:24 that $\|u - P_2 u\| \leq \varepsilon$. But $P_2 u = P_2 P u = P P_2 u$ is in PM_2 and the result follows from lemma I.1:23. □

From the preceding theorems and lemmas one can derive more composite results. The following will be used in I.2.1.

Lemma I.1:30. Suppose that

1) $M_0 \subset M_1 \oplus M_2$, M_1 and M_2 orthogonal,

2) $M_1' \oplus M_2' = \mathscr{H}$, M_1' and M_2' orthogonal,

3) $M_0 \subset_{\varepsilon'} M_1'$,

4) $M_2 \subset_{\varepsilon''} M_2'$,

Then $M_0 \subset_{\varepsilon'+\varepsilon''} M_1$

Proof. Let P_2 be the projection on M_2. By 2), 4) and theorem I.1:18 $M_1' = M_2'^c \subset_{\varepsilon''} M_2^c$. 3) and theorem I.1:16 then gives $M_0 \subset_{\varepsilon'+\varepsilon''} M_2^c$. But $M_0 \subset_0 M_1 \oplus M_2$ according to 1) and lemma I.1:27 then gives $M_0 \subset_{\varepsilon'+\varepsilon''} P_2^c(M_1 \oplus M_2) = M_1$. □

I.1.4 Approximation in Product Spaces

In this section we shall consider several complex, separable Hilbert spaces, $\mathscr{H}$, $\mathscr{H}_1$, $\mathscr{H}_2$, … simultaneously. Although we shall use the same notation $\langle\ ,\ \rangle$ for scalar product, ^{c} for orthogonal complement etc. in different spaces, there should be no risk for confusion since the arguments in the expressions will always make clear what is meant.

Definition I.1:31. Let $\mathscr{H}$, $\mathscr{H}_1$,…, $\mathscr{H}_n$, be Hilbert spaces. A mapping T of $\mathscr{H}_1 \times \ldots \times \mathscr{H}_n$ into $\mathscr{H}$ is called a tensor product mapping and we write

$$T :_{\otimes} \mathscr{H}_1 \times \ldots \times \mathscr{H}_n \to \mathscr{H}$$

if

1° $\langle T(u_1,\ldots,u_n), T(u_1',\ldots,u_n')\rangle = \langle u_1,u_1'\rangle \cdots \langle u_n,u_n'\rangle$ for any u_i, u_i' in $\mathscr{H}_i$, $i = 1,\ldots,n$

2° $\mathscr{H}$ is the closed linear hull of $T(\mathscr{H}_1 \times \ldots \times \mathscr{H}_n)$.

If M_i, $i = 1,\ldots,n$ are closed linear subspaces in $\mathscr{H}_i$, $i = 1,\ldots,n$ respectively, the closed linear hull of $T(M_1 \times \ldots \times M_n)$ will be denoted by $\bigotimes_{i=1}^{n}{}_T M_i$, or alternatively by $M_1 \otimes_T M_2 \otimes_T \ldots \otimes_T M_n$, thus especially $\mathscr{H} = \bigotimes_{i=1}^{n}{}_T \mathscr{H}_i$. If some of the factors M_i are one dimensional spanned by vectors u_i we shall also write $M_1 \otimes_T \ldots \otimes_T u_i \otimes_T \ldots$ for their product, thus especially $T(u_1,\ldots,u_n) = \bigotimes_{i=1}^{n}{}_T u_i = u_1 \otimes_T \ldots \otimes_T u_n$.

If $A_1,\ldots,A_n$ are bounded linear operators in $\mathscr{H}_1$,…, $\mathscr{H}_n$ respectively we denote by $\otimes_{1}^{n}{}_T A_i = A_1 \otimes_T \cdots \otimes_T A_n$ the bounded linear operator, which is the extension to $\mathscr{H}$ of the bounded map A_0 in $T(\mathscr{H}_1 \times \ldots \times \mathscr{H}_n)$ defined by

$$A_0\Big(\otimes_{i=1}^{n}{}_T u_i\Big) = \bigotimes_{i=1}^{n}{}_T A_i u_i, \quad u_i \text{ in } \mathscr{H}_i,\ i = 1,\ldots,n$$

If $T :_{\otimes} \mathscr{H}_1 \times \ldots \times \mathscr{H}_n \to \mathscr{H}$, we shall also say that T is a tensor product decomposition of $\mathscr{H}$ into $\mathscr{H}_1 \times \ldots \times \mathscr{H}_n$.

Lemma I.1:32. Suppose that

$$T :_{\otimes} \mathscr{H}_1 \times \ldots \times \mathscr{H}_n \to \mathscr{H}$$

and $\{u_{i,j}\}$, $j \in J_i$ is an orthonormal basis for $\mathscr{H}_i$, $i = 1, \ldots, n$. Then $\{u_k\}$, $k \in K = J_1 \times \ldots \times J_n$ where $u_k = u_{1,j_1} \otimes \ldots u_{n,j_n}$, $k = (j_1, \ldots, j_n)$, is an orthonormal basis for $\mathscr{H}$.

Proof. A simple consequence of definition I.1:31. □

Lemma I.1:33. Let $\{u_{i,j}\}$, $i \in I$, $j \in J$ be a double-indexed orthonormal basis for $\mathscr{H}$ and set $\mathscr{H}_1 = l^2(I)$, $\mathscr{H}_2 = l^2(J)$ and

$$T(u,v) = \sum_{\substack{i \in I \\ j \in J}} a_i b_j u_{i,j}$$

for $u = \{a_i\}$, $i \in I$, $v = \{b_j\}$, $j \in J$. Then

$$T :_{\otimes} \mathscr{H}_1 \times \mathscr{H}_2 \to \mathscr{H}.$$

Proof. Also a simple consequence of definition I.1:31. □

The following variant of the preceding lemma will be used in III.5.

Lemma I.1:34. Suppose that

$$\mathscr{H} = \bigoplus_{i \in I} M_i$$

is a decomposition of $\mathscr{H}$ into orthogonal closed subspaces M_i, $i \in I$, all with the same dimension and

$$U_i : \mathscr{H}_2 \to M_i$$

are isometric onto mappings and set

$$T(u,v) = \sum_{i \in I} a_i U_i v$$

for $u = \{a_i\}$, $i \in I$ in $l^2(I)$ and v in $\mathscr{H}_2$. Then $T :_{\otimes} l^2(I) \times \mathscr{H}_2 \to \mathscr{H}$.

Proof. Also a simple consequence of definition I.1:31. □

Theorem I.1:35. Suppose that $T :_{\otimes} \mathscr{H}_1 \times \mathscr{H}_2 \to \mathscr{H}$, M_1 and M_2 are closed subspaces in $\mathscr{H}_1$ and $M_1' = M_1 \otimes_T \mathscr{H}_2$, $M_2' = M_2 \otimes_T \mathscr{H}_1$. Then $M_1 \subset_\varepsilon M_2$ iff $M_1' \subset_\varepsilon M_2'$.

Proof. Let $\{u_i\}$, $i \in I = I_1 \cup I_2$ be an orthonormal basis for $\mathscr{H}_1$ such that $\{u_i\}$, $i \in I_1$ is a basis for M_2 and $\{u_i\}$, $i \in I_2$ is a basis for M_2^c. Let $\{v_j\}$, $j \in J$ be an orthonormal basis of the space $\mathscr{H}_2$. Then, by lemma I.1:32, $\{u_i \otimes_T v_j\}, i \in I$, $j \in J$ is an orthonormal basis for $\mathscr{H}$ and $\{u_i \otimes_T v_j\}$, $i \in I_k$, $j \in J$ is an orthonormal basis for $M_2 \otimes_T \mathscr{H}_2$ if $k = 1$ and for $M_2^c \otimes_T \mathscr{H}_2$ if $k = 2$.