Proof. Let \( u = \sum_{i=1}^{m} u_i, u_i \in N_i \) and \( v = \sum_{i=1}^{n} v_i, v_i \in N_i' \). Then
\[
\|u\|^2 = \sum_{i,j} \langle u_i, u_j \rangle = \sum \|u_i\|^2 + \epsilon' \quad \text{where} \quad \epsilon' = \sum_{i \neq j} \langle u_i, u_j \rangle
\]
\[
|\epsilon'| \leq \epsilon \sum_{i \neq j} \|u_i\| \cdot \|u_j\| = \epsilon \left( \left( \sum \|u_i\| \right) \left( \sum \|u_j\| \right) - \sum \|u_i\|^2 \right)
\]
\[
\leq \epsilon (m-1) \sum \|u_i\|^2.
\]
Thus
\[
\|u\|^2 \geq \left[ 1 - (m-1)\epsilon \right] \sum \|u_i\|^2 \geq \frac{1}{2} \sum \|u_i\|^2
\]
if \( 2m\epsilon < 1 \) and similarly for \( v \). Then
\[
|\langle u, v \rangle| = \left| \sum_i \sum_j \langle u_i, v_j \rangle \right| \leq \epsilon \sum_i \sum_j \|u_i\| \cdot \|v_j\| = \epsilon \left( \sum \|u_i\| \right) \left( \sum \|v_j\| \right)
\]
\[
\leq \epsilon \cdot m \cdot n \left[ \sum \|u_i\|^2 \right]^{1/2} \cdot \left[ \sum \|v_j\|^2 \right]^{1/2} \leq 2m\epsilon \cdot \|u\| \cdot \|v\|.
\]
and thus \( u \) and \( v \) are \( 2m\epsilon \)-orthogonal. If \( 2m\epsilon \geq 1 \) or \( 2n\epsilon \geq 1 \), then \( 2m\epsilon \geq 1 \) and the statement is trivially satisfied.

I.1.3 Some Auxiliary Theorems on Approxmation of Subspaces

Lemma I.1:23. \( N_1 \subset_\epsilon N_2 \) if for every vector \( u_1 \) in \( N_1 \) there exists a \( u_2 \) in \( N_2 \) with
\[
\|u_1 - u_2\| \leq \epsilon \cdot \|u_1\|
\]
Proof. A direct consequence of lemma I.1:4 and remark 1 after definition I.1.1:3 if \( N_1 \) and \( N_2 \) are closed. If \( N_1 \) or \( N_2 \) are not closed, let \( M_1 \) and \( M_2 \) be the closures of \( N_1 \) and \( N_2 \). The statement for \( N_1 \) and \( N_2 \) then follows from the statement for \( M_1 \) and \( M_2 \) by using remark 3 after theorem I.1:8.

Lemma I.1:24. \( N_1 \subset_\epsilon M_2 \) iff for every \( u \) in \( N_1 \)
\[
\|u - P_2u\| \leq \epsilon \|u\|,
\]
where \( P_2 \) is the projection on \( M_2 \)

Proof. A direct consequence of lemma I.1:4 and remark 1 after definition I.1:3

Lemma I.1:25. If \( N_1 \subset_\epsilon M_2 \) and \( P_2 \) is the projection on \( M_2 \) then
\[
N_1 \subset_\epsilon P_2 N_1 \quad \text{and} \quad P_2 N_1 \subset_\epsilon N_1.
\]

Proof. The first statement follows directly from lemma I.1:24.
Let \( u \) be an arbitrary nonzero vector in \( P_2 N_1 \). Then \( u \) is of the form \( u = P_2v \) with \( v \) in \( N_1 \) and lemma I.1:4 gives \( \text{dist}(u, v) = \text{dist}(v, u) = \text{dist}(v, P_2v) = \text{dist}(v, M_2) \leq \epsilon \).
Lemma I.1.26. Suppose that $M_1$ and $M_2$ are orthogonal and that $M_1 \subset \epsilon_1, M'_1$ and $M_2 \subset \epsilon_2, M'_2$. Then $M_1 \oplus M_2 \subset \epsilon_1 + \epsilon_2, M'_1 \oplus M'_2$.

Proof. Let $u$ be an arbitrary vector in $M_1 \oplus M_2$. Then $u$ is of the form $u = u_1 + u_2$ with $u_1$ and $u_2$ in $M_1$ and $M_2$ respectively. By lemma I.1.23 there exists $u'_1$ and $u'_2$ in $M'_1$ and $M'_2$ respectively with $\|u_1 - u'_1\| \leq \epsilon_1 \|u_1\|$ and $\|u_2 - u'_2\| \leq \epsilon_2 \|u_2\|$. Hence there exists an $u' = u'_1 + u'_2$ in $M'_1 \oplus M'_2$ with

$$\|u - u'\| \leq \|u_1 - u'_1\| + \|u_2 - u'_2\| \leq \epsilon_1 \|u_1\| + \epsilon_2 \|u_2\|$$

$$\leq (\epsilon_1^2 + \epsilon_2^2)^{1/2}(\|u_1\|^2 + \|u_2\|^2)^{1/2} = (\epsilon_1^2 + \epsilon_2^2)^{1/2}\|u\|$$

$$\leq (\epsilon_1 + \epsilon_2)\|u\|$$

since $u_1$ and $u_2$ are orthogonal and thus $M_1 \oplus M_2 \subset \epsilon_1 + \epsilon_2, M'_1 \oplus M'_2$.

Lemma I.1.27. Suppose $M \subset \epsilon', M'$ and $M \subset \epsilon'' M''$. Then $M \subset \epsilon' + \epsilon'', P''M'$ where $P''$ is the projection on $M''$.

Proof. Let $u$ be an arbitrary vector in $M$ and let $P'$ be the projection on $M'$. Then by lemma I.1.24 $\|P'u - u\| \leq \epsilon'\|u\|, \|P''u - u\| \leq \epsilon''\|u\|$ and

$$\|P''P'u - u\| \leq \|P''P'u - P''u\| + \|P''u - u\| \leq \|P''(P'u - u)\| + \epsilon''\|u\|$$

$$\leq \|P'u - u\| + \epsilon''\|u\| \leq (\epsilon' + \epsilon'')\|u\|.$$ 

Since $P''P'u$ is in $P''M'$, lemma I.1.23 then gives that $M \subset \epsilon' + \epsilon'', P''M'$. 

Lemma I.1.28. Suppose $M \subset \epsilon', M'$ and $M \subset \epsilon'' M''$. Then $P''M \subset \epsilon' + \epsilon'', M'$ where $P''$ is the projection on $M''$.

Proof. Let $u$ be a vector in $M$. Then we have that $\text{dist}(P''u, u) = \text{dist}(u, P''u) \leq \epsilon''$, $\text{dist}(u, P'u) \leq \epsilon'$ and by lemma I.1.5

$$\text{dist}(P''u, P'u) \leq \text{dist}(P''u, u) + \text{dist}(u, P'u) \leq \epsilon' + \epsilon''.$$ 

Since $P'u$ is in $M'$, this shows that $\text{dist}(P''u, M') \leq \epsilon' + \epsilon''$ and since an arbitrary vector in $P''M$ is of the form $P''u$ with $u$ in $M$, this also shows that $\text{dist}(P''M, M') \leq \epsilon' + \epsilon''$ and it follows that $P''M \subset \epsilon' + \epsilon'', M'$.

Lemma I.1.29. Suppose that $M_1 = R(P_1), M_2 = R(P_2), P$ commutes with $P_1$ and $P_2$ and $M_1 \subset \epsilon, M_2$. Then $PM_1 \subset \epsilon, PM_2$.

Proof. For arbitrary $u$ in $PM_1$, we have, since $PM_1 \subset M_1$, by lemma I.1.24 that $\|u - Pu\| \leq \epsilon$. But $Pu = P_2Pu = PP_2u$ is in $PM_2$ and the result follows from lemma I.1.23.

From the preceding theorems and lemmas one can derive more composite results. The following will be used in I.2.1.

Lemma I.1.30. Suppose that
1) \( M_0 \subset M_1 \oplus M_2 \), \( M_1 \) and \( M_2 \) orthogonal,
2) \( M_1' \oplus M_2' = \mathcal{H} \), \( M_1' \) and \( M_2' \) orthogonal,
3) \( M_0 \subset_{\epsilon'} M_1' \),
4) \( M_2 \subset_{\epsilon''} M_2' \).

Then \( M_0 \subset_{\epsilon'+\epsilon''} M_1 \).

Proof. Let \( P_2 \) be the projection on \( M_2 \). By 2), 4) and theorem I.1:18 \( M_1' = M_2'^c \subset_{\epsilon''} M_2^c \). 3) and theorem I.1:16 then gives \( M_0 \subset_{\epsilon'+\epsilon''} M_2^c \). But \( M_0 \subset_0 M_1 \oplus M_2 \) according to 1) and lemma I.1:27 then gives \( M_0 \subset_{\epsilon'+\epsilon''} P_2'(M_1 \oplus M_2) = M_1 \).

### I.1.4 Approximation in Product Spaces

In this section we shall consider several complex, separable Hilbert spaces, \( \mathcal{H}, \mathcal{H}_1, \mathcal{H}_2, \ldots \) simultaneously. Although we shall use the same notation \( \langle \ , \ \rangle \) for scalar product, \( c \) for orthogonal complement etc. in different spaces, there should be no risk for confusion since the arguments in the expressions will always make clear what is meant.

Definition I.1:31. Let \( \mathcal{H}, \mathcal{H}_1, \ldots, \mathcal{H}_n \), be Hilbert spaces. A mapping \( T \) of \( \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \) into \( \mathcal{H} \) is called a tensor product mapping and we write
\[
T : \otimes \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \rightarrow \mathcal{H}
\]
if
1* \( \langle T(u_1, \ldots, u_n), T(u'_1, \ldots, u'_n) \rangle = \langle u_1, u'_1 \rangle \cdots \langle u_n, u'_n \rangle \) for any \( u_i, u'_i \) in \( \mathcal{H}_i \), \( i = 1, \ldots, n \)
2* \( \mathcal{H} \) is the closed linear hull of \( T(\mathcal{H}_1 \times \cdots \times \mathcal{H}_n) \).

If \( M_i, i = 1, \ldots, n \) are closed linear subspaces in \( \mathcal{H}_i, i = 1, \ldots, n \) respectively, the closed linear hull of \( T(M_1 \times \cdots \times M_n) \) will be denoted by \( \bigotimes_{i=1}^n T M_i \), or alternatively by \( M_1 \otimes_T M_2 \otimes_T \cdots \otimes_T M_n \), thus especially \( \mathcal{H} = \bigotimes_{i=1}^n \mathcal{H}_i \). If some of the factors \( M_i \) are one dimensional spanned by vectors \( u_i \) we shall also write \( M_1 \otimes_T \cdots \otimes_T u_1 \otimes_T \cdots \) for their product, thus especially \( T(u_1, \ldots, u_n) = \bigotimes_{i=1}^n u_i = u_1 \otimes_T \cdots \otimes_T u_n \).

If \( A_1, \ldots, A_n \) are bounded linear operators in \( \mathcal{H}_1, \ldots, \mathcal{H}_n \) respectively we denote by \( \bigotimes_{i=1}^n T A_i = A_1 \otimes_T \cdots \otimes_T A_n \) the bounded linear operator, which is the extension to \( \mathcal{H} \) of the bounded map \( A_0 \) in \( T(\mathcal{H}_1 \times \cdots \times \mathcal{H}_n) \) defined by
\[
A_0 \left( \bigotimes_{i=1}^n u_i \right) = \bigotimes_{i=1}^n A_i u_i, \quad u_i \text{ in } \mathcal{H}_i, \ i = 1, \ldots, n
\]

If \( T : \otimes \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \rightarrow \mathcal{H} \), we shall also say that \( T \) is a tensor product decomposition of \( \mathcal{H} \) into \( \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \).
Lemma I.1:32. Suppose that
\[ T : \otimes \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \to \mathcal{H} \]
and \{u_{i,j}\}, \(j \in J_i\) is an orthonormal basis for \(\mathcal{H}_i\), \(i = 1, \ldots, n\). Then \{u_k\}, \(k \in K = J_1 \times \cdots \times J_n\) where \(u_k = u_{1,j_1} \otimes \cdots \otimes u_{n,j_n}\), \(k = (j_1, \ldots, j_n)\), is an orthonormal basis for \(\mathcal{H}\).

**Proof.** A simple consequence of definition I.1:31.

Lemma I.1:33. Let \{u_{i,j}\}, \(i \in I, j \in J\) be a double-indexed orthonormal basis for \(\mathcal{H}\) and set \(\mathcal{H}_1 = l^2(I), \mathcal{H}_2 = l^2(J)\) and
\[
T(u, v) = \sum_{i \in I} \sum_{j \in J} a_i b_j u_{i,j} \]
for \(u = \{a_i\}, i \in I, v = \{b_j\}, j \in J\). Then
\[ T : \otimes \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}. \]

**Proof.** Also a simple consequence of definition I.1:31.

The following variant of the preceding lemma will be used in III.5.

Lemma I.1:34. Suppose that
\[
\mathcal{H} = \bigoplus_{i \in I} M_i
\]
is a decomposition of \(\mathcal{H}\) into orthogonal closed subspaces \(M_i, i \in I\), all with the same dimension and
\[
U_i : \mathcal{H}_2 \to M_i
\]
are isometric onto mappings and set
\[
T(u, v) = \sum_{i \in I} a_i U_i v
\]
for \(u = \{a_i\}, i \in I\) in \(l^2(I)\) and \(v\) in \(\mathcal{H}_2\). Then \(T : \otimes l^2(I) \times \mathcal{H}_2 \to \mathcal{H}\).

**Proof.** Also a simple consequence of definition I.1:31.

Theorem I.1:35. Suppose that \(T : \otimes \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}\), \(M_1\) and \(M_2\) are closed subspaces in \(\mathcal{H}_1\) and \(M'_1 = M_1 \otimes_T \mathcal{H}_2, M'_2 = M_2 \otimes_T \mathcal{H}_1\). Then \(M_1 \subset_\varepsilon M_2\) iff \(M'_1 \subset_\varepsilon M'_2\).

**Proof.** Let \(\{u_i\}, i \in I = I_1 \cup I_2\) be an orthonormal basis for \(\mathcal{H}_1\) such that \(\{u_i\}, i \in I_1\) is a basis for \(M_2\) and \(\{u_i\}, i \in I_2\) is a basis for \(M'_2\). Let \(\{v_j\}, j \in J\) be an orthonormal basis of the space \(\mathcal{H}_2\). Then, by lemma I.1:32, \(\{u_i \otimes_T v_j\}, i \in I, j \in J\) is an orthonormal basis for \(\mathcal{H}\) and \(\{u_i \otimes_T v_j\}, i \in I_k, j \in J\) is an orthonormal basis for \(M_2 \otimes_T \mathcal{H}_2\) if \(k = 1\) and for \(M'_2 \otimes_T \mathcal{H}_2\) if \(k = 2\).