I.1.1 Distance From one Subspace to Another

Definition I.1:1. For any nonzero vectors $u$ and $v$ in the Hilbert space $H$, we define

$$\text{dist}(u, v) = (1 - |\langle u_1, v_1 \rangle|^2)^{1/2}$$

where $u_1 = u/\|u\|$ and $v_1 = v/\|v\|$.

Definition I.1:2. If $u$ is a nonzero vector and $N$ is a nonzero (not necessarily closed) subspace we define

$$\text{dist}(u, N) = \inf_{v \neq 0 \text{ in } N} \text{dist}(u, v).$$

If $u$ is nonzero and $N = 0$ we define $\text{dist}(u, N) = 1$.

Definition I.1:3. If $N_1$ and $N_2$ are nonzero subspaces, we define

$$\text{dist}(N_1, N_2) = \sup_{u \neq 0 \text{ in } N_1} \inf_{v \neq 0 \text{ in } N_2} \text{dist}(u, v).$$

If $N_1 = 0$ we define, for any $N_2$, $\text{dist}(N_1, N_2) = 0$. If $N_1$ is nonzero and $N_2 = 0$ we define $\text{dist}(N_1, N_2) = 1$.

Obviously $0 \leq \text{dist}(u, v) \leq 1$, $0 \leq \text{dist}(u, N) \leq 1$ and $0 \leq \text{dist}(N_1, N_2) \leq 1$. If $M$ is closed then $\text{dist}(N, M) = 0$ iff $N \subset M$.

Remark 1. If $M_u$ and $M_v$ are one-dimensional subspaces spanned by the vectors $u$ and $v$ respectively, we have $\text{dist}(M_u, M_v) = \text{dist}(u, M_v) = \text{dist}(u, v)$. If $N_1$ is nonzero we have $\text{dist}(N_1, N_2) = \sup_{v \neq 0 \text{ in } N_1} \text{dist}(v, N_2)$.

Remark 2. Note that $\text{dist}(u, v) = \text{dist}(v, u)$ for any (nonzero) vectors, but $\text{dist}(M_1, M_2)$ need not be equal to $\text{dist}(M_2, M_1)$. Although one gets a metric on the set of closed subspaces by defining

$$d(M_1, M_2) = \max \left( \text{dist}(M_1, M_2), \text{dist}(M_2, M_1) \right),$$

(see Kato (1) p. 198, $d(M_1, M_2) = \|P_1 - P_2\|$ by theorems I.1:7 and I.1:13 below, where $P_1$ and $P_2$ are the projections on $M_1$ and $M_2$ respectively) the “single-directed” distance $\text{dist}(M_1, M_2)$ will be important in the present theory.

Lemma I.1:4. Let $u$ be a nonzero vector, $M$ a nonzero closed subspace and $P$ the projection on $M$. Then

$$\text{dist}(u, M) = \text{dist}(u, aPu)$$

where $a$ is any complex nonzero number. If $\|u\| = 1$, then

$$\text{dist}(u, M) = \|u - Pu\|.$$  

For any vector $v$ in $M$ which is not in the form $aPu$, a complex and nonzero, we have

$$\text{dist}(u, v) > \text{dist}(u, M).$$
Proof. Let $a$ be a nonzero complex number and $v$ be any nonzero vector in $M$. Since by definitions I.1:1 and I.1:2, $\text{dist}(u,v)$ and $\text{dist}(u,M)$ does not change if we multiply by a nonzero complex number, it is no limitation to assume that $\|u\| = 1$. From definition I.1:1 then follows that $\text{dist}(u, aPu) = \|u - Pu\|$. We can choose a complex number $b$ such that $v_1 = bv$ is equal to the projection of $u$ on the one-dimensional subspace spanned by $v$. Then $v_1$ and $u-v_1$ are orthogonal and $\text{dist}(u,v) = \|u-v_1\|$. We have

$$u - v_1 = (u - Pu) + (Pu - v_1)$$

where $(u - Pu)$ and $(Pu - v_1)$ are orthogonal ($Pu - v_1$ is in $M$ and $u - Pu$ is orthogonal to $M$). Thus

$$\|u - v_1\|^2 = \|u - Pu\|^2 + \|Pu - v_1\|^2$$

and

$$\text{dist}(u,v) = \|u - v_1\| \geq \|u - Pu\| = \text{dist}(u,Pu).$$

This shows that

$$\text{dist}(u,M) = \inf_{v \neq 0 \in M} \text{dist}(u,v) = \text{dist}(u,Pu) = \text{dist}(u,aPu) = \|u - Pu\|.$$ 

If $v$ is not of the form $v = cPu$, $c$ complex and nonzero, then $\|Pu - v_1\| > 0$ and

$$\text{dist}(u,v) = \|u - v_1\| > \|u - Pu\| = \text{dist}(u,Pu).$$

Lemma I.1:5. For any nonzero $u, v, w$ in $\mathcal{H}$, we have

$$\text{dist}(u,w) \leq \text{dist}(u,v) + \text{dist}(v,w).$$

Proof. Choose $u_1 = au$, $v_1 = bv$, $w_1 = cw$, $a$, $b$, $c$ complex numbers such that $\|u_1\| = \|v_1\| = \|w_1\| = 1$, $\langle u_1, v_1 \rangle \geq 0$ and $\langle v_1, w_1 \rangle \geq 0$. Set

$$u_1 = \cos A \cdot v_1 + \sin A \cdot u'_1, \quad \cos A = \langle u_1, v_1 \rangle,$$

$$w_1 = \cos B \cdot v_1 + \sin B \cdot w'_1, \quad \cos B = \langle w_1, v_1 \rangle,$$

$$\langle u'_1, v_1 \rangle = \langle w'_1, v_1 \rangle = 0, \quad 0 \leq A, B \leq \frac{\pi}{2}.$$ 

If $A + B \leq \pi/2$

$$\langle u_1, w_1 \rangle = \cos A \cos B + \sin A \sin B \cdot d, \quad d = \langle u'_1, w'_1 \rangle, \quad |d| \leq 1$$

$$|\langle u_1, w_1 \rangle| \geq \cos A \cos B - \sin A \sin B = \cos(A + B) \geq 0.$$