I.1.1 Distance From one Subspace to Another

Definition I.1:1. For any nonzero vectors u and v in the Hilbert space $\mathscr{H}$, we define

$$\text{dist}(u,v)) = \left(1 - |\langle u_1, v_1\rangle|^2\right)^{1/2}$$

where $u_1 = u/\|u\|$ and $v_1 = v/\|v\|$.

Definition I.1:2. If u is a nonzero vector and N is a nonzero (not necessarily closed) subspace we define

$$\text{dist}(u,N) = \inf_{v\neq 0 \text{ in } N} \text{dist}(u,v).$$

If u is nonzero and $N = 0$ we define $\text{dist}(u,N) = 1$.

Definition I.1:3. If N_1 and N_2 are nonzero subspaces, we define

$$\text{dist}(N_1,N_2) = \sup_{u\neq 0 \text{ in } N_1} \inf_{v\neq 0 \text{ in } N_2} \text{dist}(u,v).$$

If $N_1 = 0$ we define, for any N_2, $\text{dist}(N_1,N_2) = 0$. If N_1 is nonzero and $N_2 = 0$ we define $\text{dist}(N_1,N_2) = 1$.

Obviously $0 \le \text{dist}(u,v) \le 1$, $0 \le \text{dist}(u,N) \le 1$ and $0 \le \text{dist}(N_1,N_2) \le 1$. If M is closed then $\text{dist}(N,M) = 0$ iff $N \subset M$.

Remark 1. If M_u and M_v are one-dimensional subspaces spanned by the vectors u and v respectively, we have $\text{dist}(M_u,M_v) = \text{dist}(u,M_v) = \text{dist}(u,v)$. If N_1 is nonzero we have $\text{dist}(N_1,N_2) = \sup_{v\neq 0 \text{ in } N_1} \text{dist}(v,N_2)$.

Remark 2. Note that $\text{dist}(u,v) = \text{dist}(v,u)$ for any (nonzero) vectors, but $\text{dist}(M_1,M_2)$ need not be equal to $\text{dist}(M_2,M_1)$. Although one gets a metric on the set of closed subspaces by defining

$$\text{d}(M_1,M_2) = \max\left(\text{dist}(M_1,M_2), \text{dist}(M_2,M_1)\right),$$

(see Kato (1) p. 198, $\text{d}(M_1,M_2) = \|P_1 - P_2\|$ by theorems I.1:7 and I.1:13 below, where P_1 and P_2 are the projections on M_1 and M_2 respectively) the "single-directed" distance $\text{dist}(M_1,M_2)$ will be important in the present theory.

Lemma I.1:4. Let u be a nonzero vector, M a nonzero closed subspace and P the projection on M. Then

$$\text{dist}(u,M) = \text{dist}(u,aPu)$$

where a is any complex nonzero number. If $\|u\| = 1$, then

$$\text{dist}(u,M) = \|u - Pu\|.$$

For any vector v in M which is not in the form aPu, a complex and nonzero, we have

$$\text{dist}(u,v) > \text{dist}(u,M).$$

Proof. Let a be a nonzero complex number and v be any nonzero vector in M. Since by definitions I.1:1 and I.1:2, $\text{dist}(u,v)$ and $\text{dist}(u,M)$ does not change if we multiply by a nonzero complex number, it is no limitation to assume that $\|u\| = 1$. From definition I.1:1 then follows that $\text{dist}(u,aPu) = \|u - Pu\|$. We can choose a complex number b such that $v_1 = bv$ is equal to the projection of u on the one-dimensional subspace spanned by v. Then v_1 and $u - v_1$ are orthogonal and $\text{dist}(u,v) = \|u - v_1\|$. We have

$$u - v_1 = \big(u - Pu\big) + \big(Pu - v_1\big)$$

where $\big(u - Pu\big)$ and $\big(Pu - v_1\big)$ are orthogonal ($Pu - v_1$ is in M and $u - Pu$ is orthogonal to M). Thus

$$\|u - v_1\|^2 = \|u - Pu\|^2 + \|Pu - v_1\|^2$$

and

$$\text{dist}(u,v) = \|u - v_1\| \geq \|u - Pu\| = \text{dist}(u,Pu).$$

This shows that

$$\begin{aligned} \text{dist}(u,M) = & \inf_{v\neq 0 \text{ in } M} \text{dist}(u,v) \\ & = \text{dist}(u,Pu) = \text{dist}(u,aPu) = \|u - Pu\|. \end{aligned}$$

If v is not of the form $v = cPu$, c complex and nonzero, then $\|Pu - v_1\| > 0$ and

$$\text{dist}(u,v) = \|u - v_1\| > \|u - Pu\| = \text{dist}(u,Pu).$$

□

Lemma I.1:5. For any nonzero u, v, w in $\mathscr{H}$, we have

$$\text{dist}(u,w) \leq \text{dist}(u,v) + \text{dist}(v,w).$$

Proof. Choose $u_1 = au$, $v_1 = bv$, $w_1 = cw$, a, b, c complex numbers such that $\|u_1\| = \|v_1\| = \|w_1\| = 1$, $\langle u_1, v_1\rangle \geq 0$ and $\langle v_1, w_1\rangle \geq 0$. Set

$$\begin{aligned} u_1 &= \cos A \cdot v_1 + \sin A \cdot u_1', \quad \cos A = \langle u_1, v_1\rangle, \\ w_1 &= \cos B \cdot v_1 + \sin B \cdot w_1', \quad \cos B = \langle w_1, v_1\rangle, \\ \langle u_1', v_1\rangle &= \langle w_1', v_1\rangle = 0, \quad 0 \leq A,\, B \leq \frac{\pi}{2}. \end{aligned}$$

If $A + B \leq \pi/2$

$$\begin{aligned} &\langle u_1, w_1\rangle = \cos A \cos B + \sin A \sin B \cdot d, \quad d = \langle u_1', w_1'\rangle, \quad |d| \leq 1 \\ &|\langle u_1, w_1\rangle| \geq \cos A \cos B - \sin A \sin B = \cos(A + B) \geq 0 \end{aligned}$$